Technische
Universität
Braunschweig

Bachelor's Thesis

New Variants of the Floodlight Problem

Yannic Lieder

August 24, 2018

**Institute of Operating Systems
and Computer Networks**

Prof. Dr. Sándor P. Fekete

Supervisors:
Dr. Christian Scheffer,
Christian Rieck, M. Sc.

Statement of Originality

This thesis has been performed independently with the support of my supervisor/s. To the best of the author's knowledge, this thesis contains no material previously published or written by another person except where due reference is made in the text.

Braunschweig, August 24, 2018

Abstract

The α–Floodlight Problem consists of finding optimal coverings of polygons with floodlights with a fixed visibility angle α. In this thesis, we introduce a variant of the floodlight problem, the Angular Art Gallery Problem (AAGP). The visibility angle of each floodlight may be set individually between 0 and 2π. The objective is not to minimize the number of floodlights, but the total visibility angle of all floodlights.

After an introduction to the AAGP, we present various bounds for different kinds of polygons. This includes an optimal covering of equilateral triangles, a tight upper bound of $(n-1)\frac{\pi}{6}$ for histograms and a general upper bound of $(n-2)\frac{\pi}{4}$ for simple polygons, if n is the number of vertices. Furthermore, we explore some other aspects of the AAGP, including a possible approach to further improve the general upper bound.

Zusammenfassung

Das α–Floodlight Problem beschäftigt sich mit der optimalen Abdeckung von Polygons mit Flutlichtern, die einen festen Öffnungswinkel α besitzen. In dieser Arbeit führen wir das Angular Art Gallery Problem (AAGP) als Variante des Flutlichtproblems ein. Hierbei ist der Öffnungswinkel nicht auf ein bestimmten Winkel α festgelegt, sondern kann für jedes Flutlicht individuell zwischen gewählt werden. Im Gegensatz zum α–Floodlight Problem ist hier das Ziel keine Minimierung der benötigten Flutlichter, sondern eine Minimierung des in der Summer aller Flutlichter benötigen Winkels.

Nach einer kurzen Einführung in das Problem zeigen wir Schranken für verschiedene Arten von Polygonen. Wir beweisen, dass gleichseitige Dreiecke mit keinem Gesamtwinkel kleiner als $\frac{\pi}{3}$ abgedeckt werden können. Wir leiten eine scharfe obere Schranke von $(n-1)\frac{\pi}{6}$ für Histogramme her und beweisen eine allgemeine obere Schranke von $(n-2)\frac{\pi}{6}$ für einfache Polygone. Außerdem betrachten wir einige andere Aspekte des AAGP, unter anderem präsentieren wir einen Ansatz, wie die allgemeine obere Schranke möglicherweise verbessert werden kann.

Aufgabenstellung

Eines der klassischen Probleme der algorithmischen Geometrie ist das sogenannte Art-Gallery-Problem (AGP). Hierbei geht es darum, einen gegebenen Grundriss mit einer möglichst geringen Anzahl von Kameras mit Rundumsicht flächendeckend zu überwachen. Das Flutlichtproblem ist eine Variante des AGP, bei der Kameras mit eingeschränktem Öffnungswinkel zu positionieren sind.

Herr Lieder soll sich im Rahmen dieser Abschlussarbeit mit neuen Varianten des Flutlichtproblems beschäftigen. Dazu gehört die Variante mit variablen Öffnungswinkeln und Ausrichtungen.

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1 Introduction

A set of well studied problems in computational geometry deals with the covering of polygons. All of these problems have their origin in the Art Gallery Problem (AGP), initially introduced by Victor Klee in 1973: Given a two-dimensional art gallery, how many watchmen are necessary to guard the art gallery completely. Translated into a mathematical model, the AGP asks for a minimum set $\mathcal{C}$ of points (guards) within a given polygon $\mathcal{P}$, such that every point $p \in \mathcal{P}$ is visible by at least one guard, i.e., for every point $p \in \mathcal{P}$ exists a $g \in \mathcal{C}$ such that the line segment $\overline{gp}$ is contained in $\mathcal{P}$.

A derivative problem concerns only guards with a limited visibility angle. While the guards in the AGP have an all around view, the guards in the α–Floodlight Problem, called floodlights, have a visibility angle, restricted to $0 < \alpha \leq 2\pi$.

In this thesis, we examine a variant of the α–Floodlight Problem. Instead of assigning a global visibility angle, it may be set individually for every floodlight. The objective is to find a set of floodlights, such that every point within a given polygon is visible by at least one floodlight and the sum of all floodlights angles is minimized. This problem is called the Angular Art Gallery Problem (AAGP).

To introduce the problem, we give a short example: Given a rectangle $\mathcal{R}$ with a height of 1 and a width of $\ell > 1$, what is the minimal total visibility angle, required to cover $\mathcal{R}$ as an instance of the AAGP? Naively, one would suspect that one floodlight with an angle of $\frac{\pi}{2}$, placed in one corner of the rectangle, is required. However, this angle may be reduced by placing two floodlights in opposite corners of $\mathcal{R}$, covering one half each (see Figure 1.1). This solution provides an overall angle smaller than $\frac{\pi}{2}$. But is this already the smallest angle, sufficient to cover $\mathcal{R}$ completely? The question remains open to this point, but gives a good impression of the non-trivially of this problem, even for very simply structured polygons.

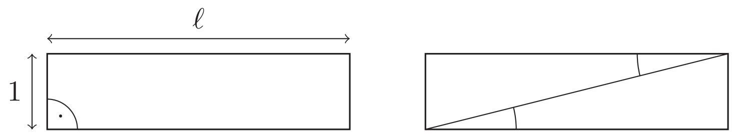


Figure 1.1. By intuition, an angle of $\frac{\pi}{2}$ is required to cover a rectangle floodlights, but this angle may be reduced by using two floodlights in opposite corners.

In this thesis, some results on upper and lower bounds for different polygon structures are presented. After a formal definition of the problem, a worst case optimal upper bound of $(n-1)\frac{\pi}{6}$, sufficient to cover any given histogram polygon with n vertices, is proven in Chapter 2. Moreover, the proof provides a polynomial algorithm to find such an upper bound covering. In Chapter 3, we prove that a minimum total angle of $\frac{\pi}{3}$ is required to cover an equilateral triangle. The third main result is an efficient algorithm, providing a covering of any simple polygon with n vertices with a total angle of at most $(n-2)\frac{\pi}{4}$. To summarize, we present the following results.

Table 1.1. Overview of the results of this thesis, the main results are framed.

Description	Lower Bound	Upper Bound	Remark
Equilateral triangles	$\boxed{\frac{\pi}{3}}$	$\frac{\pi}{3}$	
Histograms	$(n-1)\frac{\pi}{3}-\varepsilon, \varepsilon > 0$	$\boxed{(n-1)\frac{\pi}{3}}$	Uses the lower bound for equilateral triangles
Simple polygons	$(n-1)\frac{\pi}{3}-\varepsilon, \varepsilon > 0$	$\boxed{(n-2)\frac{\pi}{4}}$	Generalization of histograms

1.1. Related Work

Art Gallery Problem As previously mentioned, the family of art gallery problems covers a wide range of well studied problems. After the AGP was posed in 1973, a first result was presented in 1975 by Vasek Chvátal in the form of "Chvátal's Art Gallery Theorem": $\lfloor \frac{n}{3} \rfloor$ vertex guards are always sufficient and sometimes necessary to cover a given simple polygon with n vertices [6]. Instead of partitioning a given polygon in certain substructures as Chvátal did, Steve Fisk gave a much shorter proof of the same bound in 1978, arguing over a triangulation of the polygon and its dual graph [16]. Additionally, a tight upper bound of $\lfloor \frac{n}{4} \rfloor$ for orthogonal polygons [22, 25], i.e. polygons with all internal angles either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, and an upper bound for polygons with holes of $\left\lfloor \frac{n+2h}{3} \right\rfloor$, where h is the number of holes, has been obtained [26, pp.126–127]. Another variant is the Chromatic Art Gallery Problem (CAGP), introduced in 2010 by Erickson and LaValle [12]. Here, each guard in a guard cover is assigned a color. The CAGP asks for the minimum number of colors, required to cover a given polygon, such that no two guards with overlapping visibility polygons are assigned the same color. For CAGP, various bounds and complexity results were presented [12, 13, 15]. The NP–hardness of the fundamental AGP was proven by Lee and Lin in 1986 [23], as well as the orthogonal Art Gallery Problem [8]. It was not clear for a long time, if the AGP is in NP. In 2017, Abrahamsen et al. [1] showed that the AGP is $\exists\mathbb{R}$–complete, where $NP \subseteq \exists\mathbb{R} \subseteq PSPACE$.

α–Floodlight Problem Some results regarding the α–Floodlight Problem refer to the covering of polygons with a limited number of floodlights. In [14], an $\mathcal{O}(n^2)$ time algorithm is presented to determine a set of two floodlights, covering a convex polygon with n vertices, such that the sum of their angles is minimized. In [28, pp. 981–982], it is proven that any convex polygon can be covered by three vertex floodlights with given angles $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. A similar result is given in [20]: Any convex polygon with $n \geq 4$ vertices can be covered by four $\frac{\pi}{4}$–floodlights. This is not generally true: Given a polygon with n vertices and $k \leq n$ α–floodlights with $\alpha = \frac{\pi}{k}$, the floodlights cannot be assigned to distinct vertices and cover the polygon completely in general [27], except the polygon is circular [17], i.e. all vertices lie on a circle. Another similar result is proven in [7]: Three $\frac{\pi}{6}$– and three $\frac{\pi}{4}$–floodlights are sufficient to cover any triangle and any convex quadrangle, respectively. Moreover, the α–Floodlight Problem is shown to be NP–hard for any $0 < \alpha \leq 2\pi$ by Bagga [3].

Other related problems The Stage Illumination Problem is defined as follows: Given a line segment (stage) L on the x axis and k floodlights with visibility angles $\alpha_1, \ldots, \alpha_k$, placed anywhere in the plane above the x axis, is there a rotation of the floodlights, such that L is completely covered. This problem is proven to be NP–hard [21]. A variant of this problem takes L and a set of points P as instance and asks for a minimum floodlight covering with floodlights, placed on points in P only. In this context, minimum means a minimum total floodlight angle. For this variant, an $\mathcal{O}(n \log n)$ time algorithm was presented by Czyzowicz et al. [9].

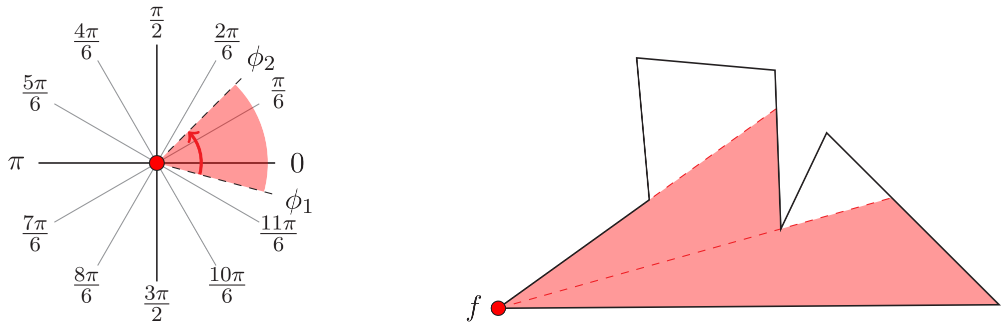
Another related problem is the (Altitude) Terrain Guarding Problem ((A)TGP). Given an x-monotone polygonal chain (terrain), find a minimum set of guards, placed on the vertices of the terrain, such that the whole terrain is covered. The TGP is shown to be NP–hard, but there are various approximation algorithms known, including a 4-approximation [10] and a polynomial-time approximation scheme [19]. The ATGP asks not for a guard set with guards placed on the vertices of the terrain, but an optimal covering with guards, placed on an altitude line. This structure has strong similarities to the histograms, considered in this thesis. For the ATGP, a polynomial time algorithm is known, solving it optimal [18].

A completely different concept of floodlights is used in a distantly related problem, which coincidentally is also referred to the Floodlight Problem [4]: Given three angles $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$, n points in the plane and three integral numbers $k_1 + k_2 + k_3 = n$, there is always a tripartition of the plane into three wedges, such that the i-th wedge has an angle of α_i and contains k_i points. Such a tripartition can be determined in $\mathcal{O}(n \log n)$.

1.2. Problem Definition

Initially, we give some preliminaries to the Art Gallery Problem. Given a polygon $\mathcal{P}$, a *guard* g is defined as a point within $\mathcal{P}$ and it *sees* another point $p \in \mathcal{P}$, if and only if the line segment $\overline{gp}$ is contained in $\mathcal{P}$. We also refer to p as being *visible* from g. We distinguish between two kinds of guards: While *point guards* may be located on any point within $\mathcal{P}$, *vertex guards* are restricted to being placed on the vertices of $\mathcal{P}$. The *visibility polygon* of g consists of all points that are visible by g. A point $p \in \mathcal{P}$ is *covered*, if and only if it is visible by at least one guard. $\mathcal{P}$ is covered, if and only if every point $p \in \mathcal{P}$ is covered.

In contrast to the AGP, we consider floodlights instead of guards in the Angular Art Gallery Problem. A floodlight is, like a guard, placed on a point within the polygon. In addition, it is defined by two angles ϕ_1 and ϕ_2, which define the *visibility cone* of the floodlight and restrict the visibility polygon.



(a) The floodlight is defined by $\left(p, \frac{23\pi}{12}, \frac{\pi}{4}\right)$, where the angles are understood in the polar coordinate system, centered by p.

(b) The red area is the visibility polygon of the floodlight f.

Figure 1.2. Exemplary guard definition and visibility

Definition 1 (Floodlight)**.** A floodlight is a tuple $f = (p, \phi_1, \phi_2)$, where $p \in \mathcal{P}$ is a point within a given polygon $\mathcal{P}$ and the angles $0 \leq \phi_1, \phi_2 < 2\pi$ define the *visibility cone* of f. The angles are understood as polar angles in the domain $[0, 2\pi]$ with p as pole. The cone is spanned counterclockwise from ϕ_1 to ϕ_2 (see Figure 1.2a). The angle of f is $\measuredangle_f = (\phi_2 - \phi_1) \mod 2\pi$.

Definition 2 (Visibility)**.** Consider a floodlight $f = (p, \phi_1, \phi_2)$ within a polygon $\mathcal{P}$. It *covers* a point $q \in \mathcal{P}$, if and only if the line segment $\overline{pq}$ is completely within both, $\mathcal{P}$ and the visibility cone of f. The *visibility polygon* of f is the polygon defined by $\{q \in \mathcal{P} \mid q \text{ is covered by } f\}$ (see Figure 1.2b).

Definition 3 (Covering)**.** Given a simple polygon $\mathcal{P}$, a covering $\mathcal{C}$ is a set of floodlights within $\mathcal{P}$. $\mathcal{C}$ is *feasible*, if and only if it covers $\mathcal{P}$ completely, i.e., every point $p \in \mathcal{P}$ is covered by at least one floodlight $f \in \mathcal{C}$.

Definition 4 (Total floodlight angle)**.** Given a covering $\mathcal{C}$, the *total floodlight angle* is lifted from the individual floodlight angles as

$$\measuredangle_{\mathcal{C}} = \sum_{f \in \mathcal{C}} \measuredangle_f .$$

The objective in the AAGP is not to minimize the number of floodlights, but to minimize the total floodlight angle. Formally, it may be defined as:

<u>Angular Art Gallery Problem (AAGP)</u>

Given: A polygon $\mathcal{P}$

Wanted: A covering $\mathcal{C}$, feasible wrt. $\mathcal{P}$ with $\measuredangle_{\mathcal{C}}$ minimized

2 Upper Bound for Histogram Polygons

It is generally assumed that a total floodlight angle of $(n-1)\frac{\pi}{6}$ is always sufficient to cover any polygon with n vertices completely. In this chapter, we present a constructive proof, confirming this upper bound for histogram polygons.

Definition 5 (Histogram)**.** A histogram is a polygon $\mathcal{P}$ with n vertices, where one edge functions as the *baseline* s. The line segment orthogonal to s between any vertex v and s is within $\mathcal{P}$. The other $n-1$ edges are called *peak edges*.

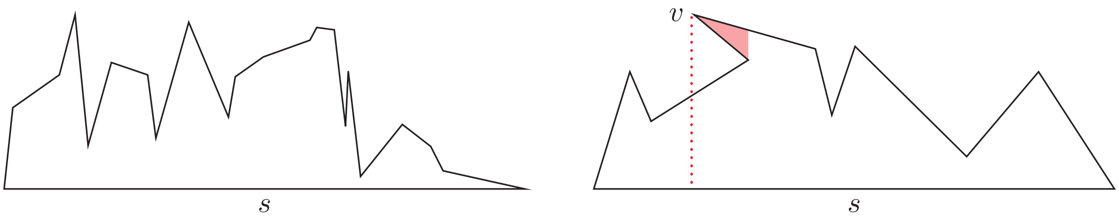


Figure 2.1. Only the left polygon is a histogram because in the right one the line segment orthogonal to s between v and s is not within the polygon.

2.1. Covering Algorithm

We present an algorithm, providing for a given histogram with n vertices a set of floodlights, which covers the histogram with a total floodlight angle of at most $(n-1)\frac{\pi}{6}$. Intuitively, the algorithm explores the histogram from left to right and use an angle of at most $\frac{\pi}{6}$ to cover each peak edge. Later, we will show that the whole polygon is covered, when all peak edges are covered.

Initially, we fix some details of the examined polygon. We assume that the baseline s lies on the x axis and the peak edges are located above s. In the following, a histogram is labeled as shown in Figure 2.2: The vertices as well as the peak edges are numbered ascending from left to right, starting with 0.

Furthermore, we define the *left* and *right visibility extension* of a peak edge, which is used several times.

Definition 6 (Left and right visibility extension)**.** Let e be a peak edge in a histogram.

1. e is called an *upward edge*, if the distance between the left vertex of e and s is strictly lower than the distance between right one and s. Otherwise, it is called a *downward edge*.

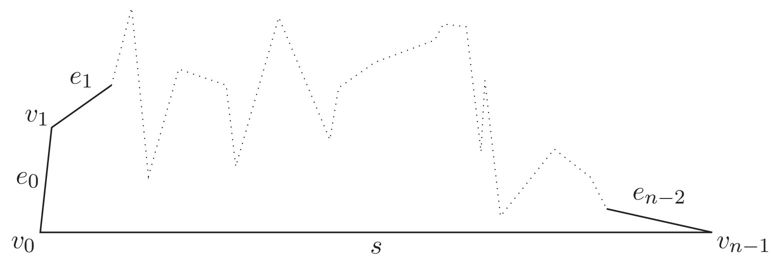


Figure 2.2. The edges and vertices of a histogram are numbered ascending from the left to the right. The baseline is labeled as s.

2. The *left visibility extension* $VE_{\text{left}}(e)$ of e is the ray that extends e to the left. If e is an upward edge, the left visibility extension extends e downwards, otherwise upwards.

3. The *right visibility extension* $VE_{\text{right}}(e)$ of e is the ray that extends e to the right.

4. In the special case that e is orthogonal to s, both, $VE_{\text{left}}(e)$ and $VE_{\text{right}}(e)$ are the rays, extending e downwards.

Since $VE_{\text{left}}(e)$ and $VE_{\text{right}}(e)$ are defined as rays, they may intersect with more than one edge of the polygon. In the following, we define an intersection of $VE_{\text{left}}(e)$ and $VE_{\text{right}}(e)$ with the polygon as follows: If e is orthogonal to s, $VE_{\text{left}}(e) = VE_{\text{right}}(e)$ intersects with s. Otherwise, consider the vertx v_l, which forms the left endpoint of e. If v_l is convex, the intersection of $VE_{\text{left}}(e)$ with the polygon is v_l. If v_l is reflex, the intersection of $VE_{\text{left}}(e)$ is defined as the intersection with the polygon that is nearest to v_l, but not v_l itself. The intersection of $VE_{\text{right}}(e)$ is defined analogously.

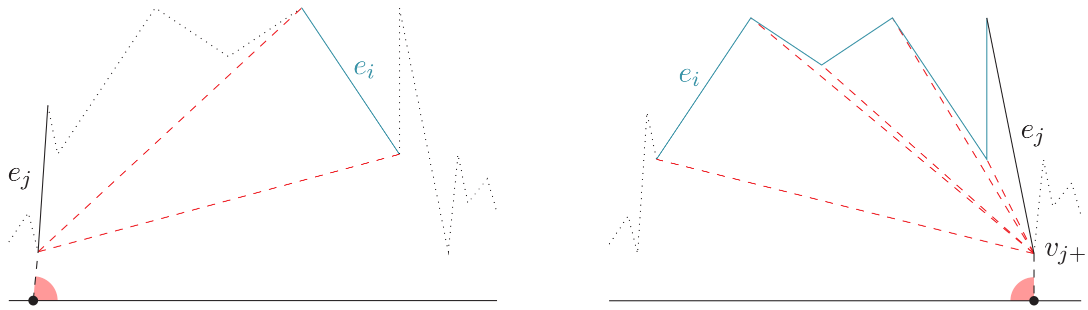
Starting with e_0, the algorithm iterates over the peak edges from left to right and treats six different cases in every iteration. Three different kinds of floodlights will be placed during this process: *Forward* and *backward covering floodlights*, which are floodlights placed on the baseline s and cover the polygon with an angle of at most $\frac{\pi}{2}$ to the right and to the left, respectively. Furthermore, there are *triangle floodlights*, placed in a defined triangle on the corner with the smallest internal angle, covering the triangle.

In the following, we give an informal description of the six cases the algorithm distinguishes when reaching a new edge. The cases have an hierarchical structure, i.e. a case may apply to an edge, only if all preceding cases do not hold. For each case is a condition and a floodlight placement action given. A formal definition can be found in Appendix A. Let e_i be the currently considered edge.

Case 1

Condition: e_i is completely covered by a triangle floodlight, by a forward covering floodlight placed below an edge $e_j, j < i$ and e_i is visible from v_j (see Figure 2.3a), or by a backward covering floodlight placed below an edge $e_j, j > i$ and all edges between e_i and e_j are visible from v_{j+1} (see Figure 2.3b).

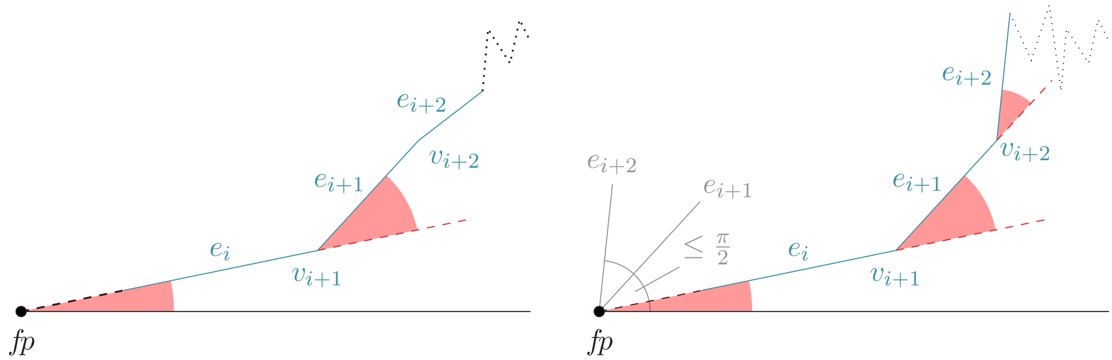
Placement: No floodlight is placed in this case.



(a) e_i is visible from v_j and a forward covering floodlight is placed below e_j.

(b) A backward covering floodlight is placed below e_j, e_i is visible from v_{j+1} and all edges between e_i and the floodlight are visible from v_j, too.

Figure 2.3. Case 1: e_i is already covered.



(a) The second vertex convex. By placing floodlights on fp and v_{i+1}, the overall angle is at most $\frac{\pi}{2}$ and the floodlights cover at least three edges.

(b) The second vertex is reflex, too. By placing floodlights on fp, v_{i+1} and v_{i+2}, the overall angle is at most $\frac{\pi}{2}$ and the floodlights cover at least three edges.

Figure 2.4. Case 2: The vertex v_{i+1} is reflex.

If Case 1 does not apply to e_i, $VE_{\text{left}}(e_i)$ intersects with s. We select this intersection point, called *fp*, as a possible location to place a forward covering floodlight.

Case 2

Condition: The vertex v_{i+1} is reflex (see Figure 2.4).

Placement: A forward covering floodlight is placed on *fp* and its visibility cone gets expanded by placing a second guard on v_{i+1}, as visualized in Figure 2.4a. If v_{i+2} is convex, the first three edges are covered with an overall angle of at most $\frac{\pi}{2}$. Otherwise, the visibility cone gets expanded a second time by placing a third floodlight on v_{i+2} (see Figure 2.4b). The overall angle is still at most $\frac{\pi}{2}$, since the angle between the first and the last edge in a sequence of edges with reflex vertices is at most $\frac{\pi}{2}$ in a histogram. At least these three first edges are covered.

When neither Case 1, nor Case 2 applies to e_i, the vertex v_{i+1} is convex. Hence, the edge e_{i+1} is visible from *fp* and a floodlight on *fp* covers at least e_i and e_{i+1}.

Case 3

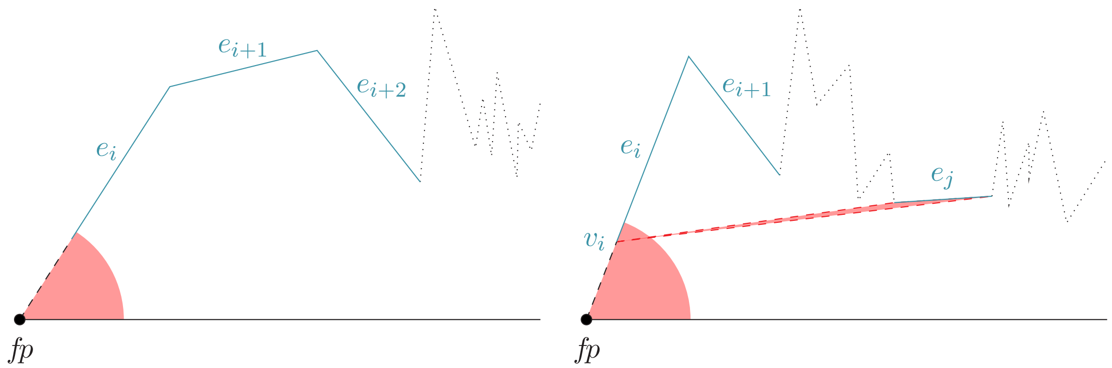
Condition: The second vertex v_{i+2} behind e_i is convex, too.

Placement: A floodlight with an angle of at most $\frac{\pi}{2}$ is placed on *fp* and covers at least the three edges e_i, e_{i+1} and e_{i+2} (see Figure 2.5).

Case 4

Condition: The second vertex v_{i+2} is reflex, but there is at least one edge e_j with $j > i+1$, which is visible from v_i and not covered as required in Case 1. Furthermore, v_j is reflex (see Figure 2.5).

Placement: A forward covering floodlight on *fp* with an angle of at most $\frac{\pi}{2}$ covers at least e_i, e_{i+1} and e_j.



(a) Case 3: Since v_{i+1} and v_{i+2} are convex, the edges e_i, e_{i+1} and e_{i+2} can be covered by a floodlight on *fp*.

(b) Case 4: There is an edge e_j with $j > i+1$, such that v_j is reflex, e_j is visible from v_i and e_j is not covered, yet.

Figure 2.5. Cases 3 and 4

The last two cases occur only, if there is no edge with these properties. It may happen that a forward covering floodlight is not longer sufficient to cover edges with an angle of at most $\frac{\pi}{6}$ per edge. Hence, in the cases 5 and 6, the other two kinds of floodlights are used.

Case 5

Condition: The right visibility extension $VE_{\text{right}}(e_{i+1})$ of the second edge e_{i+1} intersects with s.

Placement: The visibility extensions of e_i and e_{i+1} form a triangle with some part of s (see Figure 2.6). A floodlight is placed on the corner of the triangle with the smallest internal angle, which is at most $\frac{\pi}{3}$. It covers the whole triangle, especially e_i and e_{i+1}.

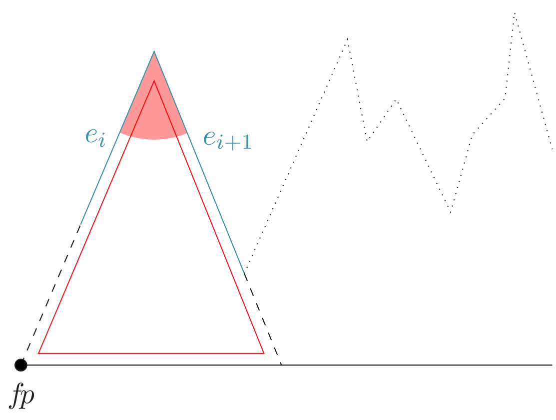
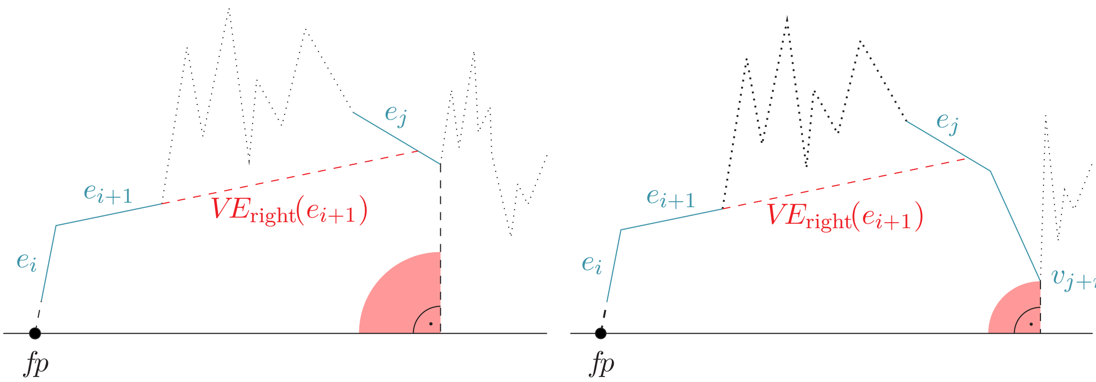


Figure 2.6. Case 5: The visibility extensions of e_i and e_{i+1} intersects with s. A triangle floodlight with an angle of at most $\frac{\pi}{3}$ is placed in the corner with the smallest internal angle of the resulting triangle.

Case 6

Condition: If none of the cases 1–5 hold, case 6 applies. The right visibility extension $VE_{\text{right}}(e_{i+1})$ of e_{i+1} does not intersect with s.

Placement: Let e_j with $j > i + 1$ be the intersection edge of $VE_{\text{right}}(e_{i+1})$ and the histogram. Let $v_{j+r}, r > 0$ be the first reflex vertex after v_j. Placing a backward covering $\frac{\pi}{2}$ – floodlight on s at the x coordinate of v_{j+r}, at least e_i, e_{i+1}, $e_j, \ldots, e_{j+r-1}$ are covered (see Figure 2.7). In the special case that the same backward covering floodlight was already placed earlier and the edge e_j is already used, skip the placement, because e_i is already covered by this floodlight. In case 6, we have to skip the next edge e_{i+1}, too, which is covered by the same floodlight as e_i.



(a) The right visibility extension of e_{i+1} intersects with an edge $e_j, j > i + 1$. A backward covering $\frac{\pi}{2}$ – floodlight covers at least e_i, e_{i+1} and e_j.

(b) If the vertex next after e_j is convex, the floodlight gets shifted until the first reflex vertex occurs. Then, the edges between e_j and the floodlight are covered, too.

Figure 2.7. Case 6: Backward covering floodlight

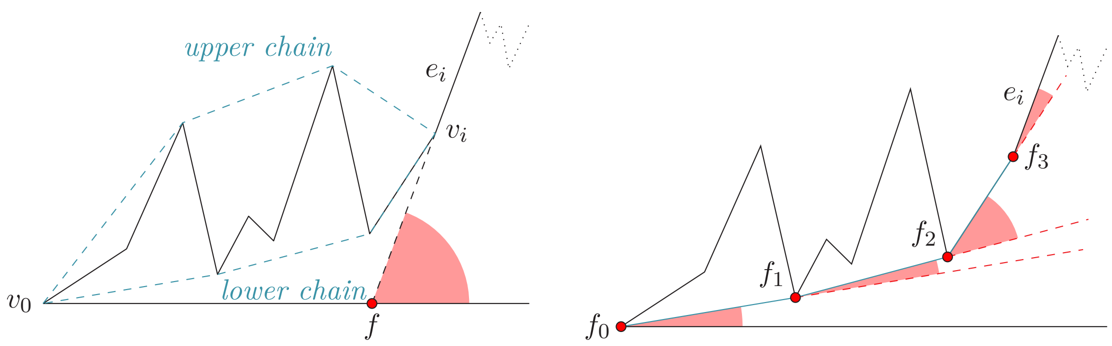
2.2. Adaption to Vertex Floodlights

Until now, the algorithm provides a covering with point floodlights only, since floodlights may be placed on the edge s. In the following, a variant of the algorithm is presented, which provides a covering with vertex floodlights only.

Consider a feasible covering as provided by the algorithm. The floodlights placed on a vertex do not change. The remaining floodlights are placed on s. To replace them by vertex floodlights, we make use of the convex hull of a subset of the polygon vertices.

Definition 7 (Convex hull)**.** The *convex hull* of a finite set of points P is the smallest convex polygon that includes all $p \in P$. The vertices of the convex hull are a subset of P.

Consider a forward covering floodlight f, placed on the intersection of s and the left visibility extension of an edge e_i. The convex hull of the vertices $v_0, \dots, v_i$ consists of two vertex chains, an upper and a lower vertex chain, both starting with v_0 and ending with v_i (see Figure 2.8a). We consider the lower chain that is closer to s. The considered floodlight with an angle $\measuredangle_f$ can be split to a set of floodlights $f_0, \dots, f_k$, located on the vertices of the lower chain of the convex hull, whose overall angle is $\measuredangle_{f_0} + \cdots + \measuredangle_{f_k} = \measuredangle_f$ (see Figure 2.8b). Analogously, the same procedure works for backward covering floodlights, placed on the x coordinate of the right vertex of e_i with the convex hull of $v_{i+1}, \dots, v_{n-2}$. Since we always use $\frac{\pi}{2}$–floodlights for backward covering, $\measuredangle_{f_0} + \cdots + \measuredangle_{f_k} \leq \measuredangle_f$.



(a) The convex hull of the vertices $v_1, \dots, v_i$ consists of two vertex chains, an upper and a lower chain.

(b) By placing floodlights on all vertices of the lower convex hull chain, the point guard can be replaced by vertex floodlights with the same overall angle.

Figure 2.8. Example of point guard to vertex guard conversion.

However, this method of point floodlight replacement has some limitations in terms of visibility. In general, the replacing vertex floodlights do not cover the same area as the replaced point floodlights. Consider Figure 2.9. Both edges $\tilde{e}$ and $\tilde{\tilde{e}}$ are completely covered by the floodlight f, but after the conversion to vertex floodlights, neither is visible from one of the vertex floodlights. This results from the fact that none of the two edges are visible from the vertices of the lower convex hull chain. However, this problem is counteracted by the design of the original point floodlight algorithm. It ensures that a

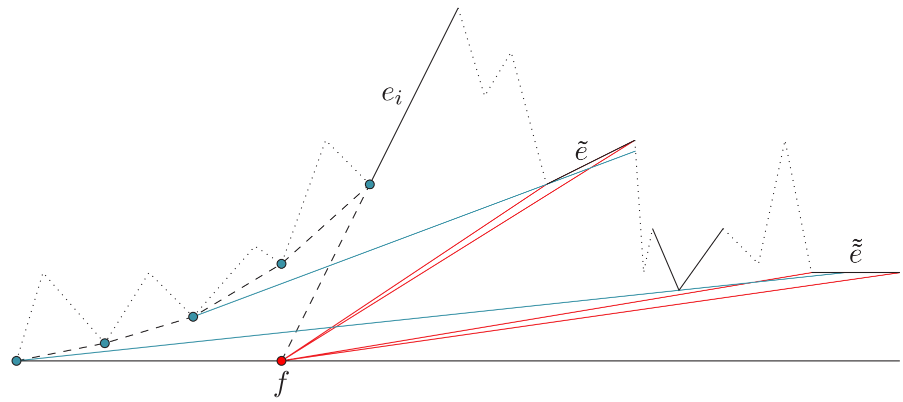


Figure 2.9. The edges $\tilde{e}$ and $\tilde{\tilde{e}}$ are visible from the point floodlight f, but not by its associated vertex floodlights.

floodlight is only placed whenever the used edges are visible from the highest point of the lower convex hull chain. As a result, an edge is covered by a point floodlight, if and only if it is covered by the associated vertex floodlights, too.

2.3. Correctness and Time Complexity

In this section, the correctness of the presented algorithm is proven and an analysis of its time complexity is given.

Correctness

Theorem 1. *Given a histogram $\mathcal{P}$ with n vertices, the algorithm, presented in Section 2.1, provides a feasible covering $\mathcal{C}$ with at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ point floodlights and $\measuredangle_{\mathcal{C}} \leq (n-1)\frac{\pi}{6}$.*

Proof. We distinguish the six cases, summarized in Table 2.1. The algorithm treats one peak edge in every iteration. Hence, it terminates after $n-1$ iterations. In the following, we call an edge *used*, if a floodlight uses an angle of at most $\frac{\pi}{6}$ with the argument that it covers this edge. Hence, it have to be shown that no edge is used by more than one floodlight to satisfy the bound. As an example, a floodlight placed in Case 3 uses the edges e_i, e_{i+1} and e_{i+2} and a floodlight placed in Case 5 uses only the edges e_i and e_{i+1}. It can be seen that a floodlight placed in any case has an angle of at most $x\frac{\pi}{6}, x \in \{2, 3\}$ and uses and covers x edges.

Next, we prove that all edges e_j with $j < i$ are covered, when the algorithm reaches an edge e_i with $0 \leq i < n-1$ by induction.

Consider $i = 0$. Trivially, $VE_{\text{left}}(e_0)$ intersects with s and no edge is used, yet. When none of the first 5 cases apply to e_0, the histogram has the following structure: v_1 is convex and v_2 is reflex, $VE_{\text{right}}(e_1)$ intersects not with s, but with a peak edge $e_j, j > 2$. This is exactly the configuration treated by case 6.

Assume the statement holds for an arbitrary edge e_{i-1} with $0 < i < n-1$. Consider the edge e_i. All preceding edges $e_0, \ldots, e_{i-1}$ are already covered by induction hypothesis.

Table 2.1. The six disjoint cases, the algorithm handles. A case includes the negations of all previous cases.

Case	Condition
1	e_i is covered in a special way
2	v_{i+1} is reflex
3	v_{i+1} and v_{i+2} are convex
4	$\exists j : j > i + 1 : e_j$ is not covered as required in Case 1, e_j is visible from v_i and v_j is reflex
5	$VE_{\text{right}}(e_{i+1})$ intersects with s
6	None of the above applies

If Case 1 applies to e_i, it is already covered and will be skipped. As such, we assume that Case 1 does not apply. We treat the remaining possibilities.

At first, we show that $VE_{\text{left}}(e_i)$ intersects with s. We show that this constellation cannot occur by deriving a contradiction. Assume $VE_{\text{left}}(e_i)$ intersects with a peak edge $e_k, k < i$. By induction hypothesis, e_k is already covered. Consider the different kinds of floodlights that may cover e_k:

(1) *e_k is covered by a forward covering floodlight:*
A forward covering floodlight would also cover e_i, since there cannot be an edge located between $VE_{\text{left}}(e_i)$ and s and restrict the visibility of e_i to the floodlight (see Figure 2.10a).

(2) *e_k is covered by a triangle floodlight:*
Triangle floodlights are exclusively placed in Case 5. Since $VE_{\text{left}}(e_i)$ intersects with e_k, e_k has to be the left edge of the triangle and the floodlight has been placed in iteration k. If v_i is convex, $e_i = e_{k+1}$ holds and e_i is the right edge of the triangle. Hence, v_i has to be reflex. With the same argument as in (1), the edge e_i is visible from v_k. This also fulfills the criteria of Case 4 This is a valid configuration for Case 4, which would have been handled prior (see Figure 2.10b). Hence, e_k cannot be covered by a triangle floodlight.

(3) *e_k is covered by a backward covering floodlight f:*
Consider a backward covering floodlight f that covers e_k. We distinguish two cases:

(a) *f was placed in iteration k (including the special case in Case 6 that the floodlight is already placed):*
Such a floodlight would have been placed exclusively in Case 6. If $e_i = e_{k+1}$, e_i would have been skipped as the special case in Case 6. Otherwise, v_i is reflex and the same arguments as in (2) hold.

(b) *f was placed in an earlier iteration:*
Assume f was placed in an earlier iteration $l < k$ and is placed below an edge $e_j, j \geq k$.

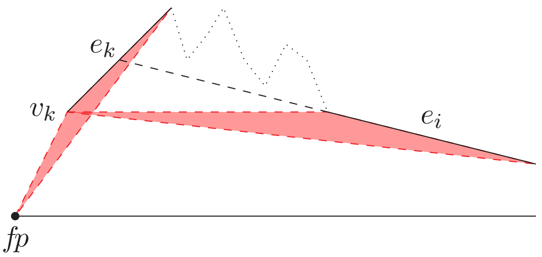
- *f lies between e_k and e_i ($k \leq j < i$):*
  In this constellation, the edge e_i is visible from the vertex v_l (see Figure 2.10c). If v_i is reflex, e_l would have been treated in Case 4 and a forward covering floodlight, that covers e_l, e_{l+1} and e_i, would have been placed instead of the backward covering floodlight. Otherwise, v_i is convex. Then, $e_j = e_k = e_{i-1}$. This constellation is treated in Case 6 and the backward covering floodlight would not have been placed below e_j, but below the next reflex vertex, which is to the right of e_i. Hence, such a backward covering floodlight cannot occur.
- *f lies to the right of e_i ($j > i$):*
  To simplify this proof step, we assume that $VE_{\text{left}}(e_k)$ intersects with s (see Figure 2.10d). By successive applying this proof step to the edge, $VE_{\text{left}}(e_k)$ intersects with, the result holds even, if $VE_{\text{left}}(e_k)$ intersects with another peak edge. The edge e_k is only skipped in Case 1, if all edges between e_k and the backward covering floodlight are visible from the vertex above the floodlight, too. If this condition is fulfilled, all edges between e_i and the floodlight, including e_i, are visible by this floodlight, too. Then, Case 1 would have been applied to e_i, which is a contradiction. Hence, e_k cannot be skipped in Case 1 with the argument of the backward covering floodlight. The only exception is the special case in Case 6, but this is already treated above.

Altogether, when Case 1 does not apply to an edge e_i, $VE_{\text{left}}(e_i)$ always intersects with s.

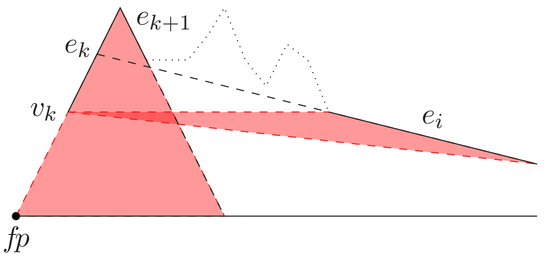
In all remaining cases, the edge e_{i+1} right after e_i is used for covering. Next, we prove that e_{i+1} is not previously used. Assume e_{i+1} is already used. The only configurations, where an edge to the right of the current considered one is already used, occur in Case 4 or 6. If e_i is covered by a forward covering floodlight in Case 4, this floodlight would also cover e_i and would apply to Case 1, regardless of whether v_{i+1} is reflex or convex. If e_{i+1} is used by a backward covering floodlight in Case 6 and v_{i+1} is convex, e_i would apply to Case 1, too. Since e_i is, as previously reasoned, an upward edge, it would be covered by the backward covering floodlight and would apply to Case 1, even if v_{i+1} is reflex. Consider the remaining possibly occurring configurations:

(1) *v_{i+1} is reflex:*
This configuration fits to of Case 2. A forward covering floodlight is placed to cover e_i, e_{i+1} and e_{i+2}, and uses these edges, too. e_{i+2} is not used yet, by the same arguments as e_{i+1}: Neither a forward, nor a backward covering floodlight can cover e_{i+2} completely without covering e_i, too.

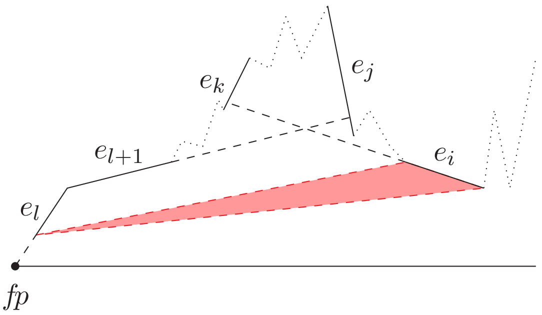
(2) *v_{i+1} is convex:*
   (a) *v_{i+2} is convex:*
   This is a configuration as required in Case 3. A forward covering floodlight is placed to cover the edges e_i, e_{i+1} and e_{i+2}. e_{i+2} is not used yet: A backward covering floodlight (Case 6), covering e_{i+2}, would also cover e_i. Furthermore,



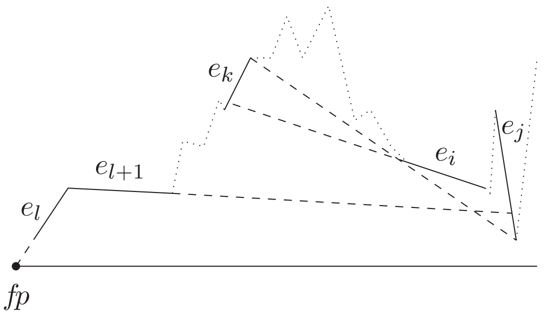
(a) If e_k is covered by a forward covering floodlight, this floodlight covers e_i, too.



(b) e_k cannot be covered by a triangle floodlight, since e_k would prior handled in Case 5 with $e_j = e_i$.



(c) A backward covering floodlight between e_k and e_i cannot occur.



(d) A backward covering floodlight, placed to the right of e_i and covers e_k, will be ignored.

Figure 2.10. An edge e_i, where $VE_{\text{left}}(e_i)$ does not intersect with s, but with a peak edge e_k, is already covered.

e_{i+2} is not used in Case 4 (forward covering floodlight), since there is the condition that the preceding vertex v_{i+2} is reflex (see Figure 2.11).

(b) *v_{i+2} is reflex:*

- *There is an edge $e_j, j > i+1$ that is visible from v_i, v_j is reflex and it is not covered as required in Case 1:*
  This is the condition of Case 4. A forward covering floodlight covers e_i, e_{i+1} and e_j. e_j is not used, since otherwise, it would also be covered as required in Case 1.
- *There is no such edge and $VE_{\text{right}}(e_{i+2})$ intersects with s:*
  This fits to Case 5 and a triangle floodlight is placed and covers the edges e_i and e_{i+1}.
- *There is no such edge and $VE_{\text{right}}(e_{i+2})$ does not intersect with s, but with another peak edge $e_j, j > i+2$:*
  This remaining configuration is treated by Case 6. The edge e_j may only be used by a backward covering floodlight, placed in Case 6 earlier. Then, e_i would have been skipped without floodlight placing, since this would be the special case in Case 6.

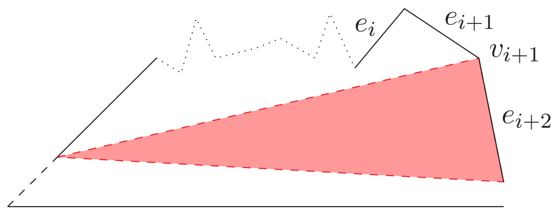


Figure 2.11. In this configuration, the edge e_{i+2} is not treated in Case 4, since the preceding vertex v_{i+2} is not reflex.

Because the preceding case analysis is exhaustive, one of the six cases applies to the peak edge e_i. The edge e_i is always covered after the execution of one of the cases and it will be proceeded with e_{i+1}. In Case 6, e_{i+1} is covered, too and the next considered edge is e_{i+2}.

Finally, it remains to show that the determined covering is feasible. This follows due to two properties of the algorithm. On the one hand, the floodlights are placed in a way that each peak edge is covered. On the other hand, we have three kinds of floodlights: The forward and the backward covering floodlights are placed on the baseline and cover the areas between the baseline and all peak edges, covered by this floodlight. The third kind are the triangle floodlights. These floodlights cover the whole triangle, especially the whole area between the baseline and the two peak edges. Hence, the area under a peak edge e is covered, if e is covered. Because all edges are covered when the algorithm terminates, the whole polygon is covered, too.

Since an angle of at most $\frac{\pi}{6}$ is used to cover each of the $n-1$ peak edges and at least one edge is skipped, every time we place a floodlight, we obtain a total floodlight angle of at most $(n-1)\frac{\pi}{6}$ and a maximum number of floodlights of $\left\lfloor \frac{n-1}{2} \right\rfloor$. □

Time Complexity

Lemma 1. *The time complexity of the algorithm is* $\mathcal{O}(n^2)$.

Proof. In a preprocessing step, a data structure which maps all polygon edges to the intersections of their visibility extensions can be determined in $\mathcal{O}(n^2)$. Using a suitable data structure, the access to the intersection edges can be realized in asymptotic constant time.

The algorithm iterates at most once over the $n-1$ peak edges. In each iteration, one of the six cases is handled. The Cases 2, 3 and 5 are treated in constant time. Case 6 takes $\mathcal{O}(n)$ time at finding the first reflex vertex right of e_j. In the remaining Cases 1 and 4, the coverage of some edges by floodlights has to be checked. Checking and storing the covered edges every time a new floodlight is placed can be done in $\mathcal{O}(n)$ time. Then, the Cases 1 and 4 can be treated in constant time. The time complexity of the remaining cases, where perhaps a floodlight is placed, increases by $\mathcal{O}(n)$. The maximum of all cases is still $\mathcal{O}(n)$ with this modification.

Hence, the algorithm requires $\mathcal{O}(n^2)$ time in the worst case. □

2.4. Worst Case Optimality

Consider the polygon, given in Figure 2.12. By decreasing $d \to 0$, we obtain $\frac{n-1}{2}$ subpolygons, which have almost the shape of an equilateral triangle. The area in such a subpolygon, visible from more than the three corresponding triangle vertices, is nearly 0. Hence, in an optimal AAGP solution, each of these triangles get covered effectively independently. Let ε be the overall angle, saved by covering the lower part of the triangles not independently of each other. This angle approaches 0, when decreasing $d \to 0$. In Chapter 3, we show that a total floodlight angle of $\frac{\pi}{3}$ is required to cover an equilateral triangle. This yields a total floodlight angle of the considered polygon of at least

$$\frac{n-1}{2}\frac{\pi}{3} - \varepsilon = (n-1)\frac{\pi}{6} - \varepsilon.$$

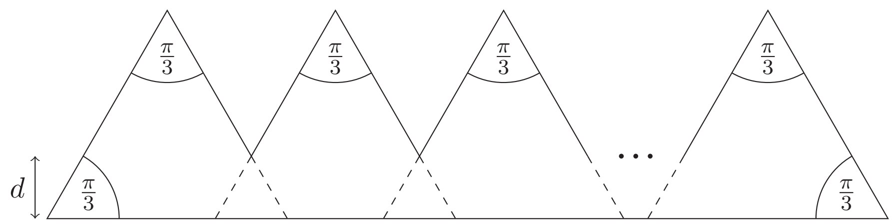
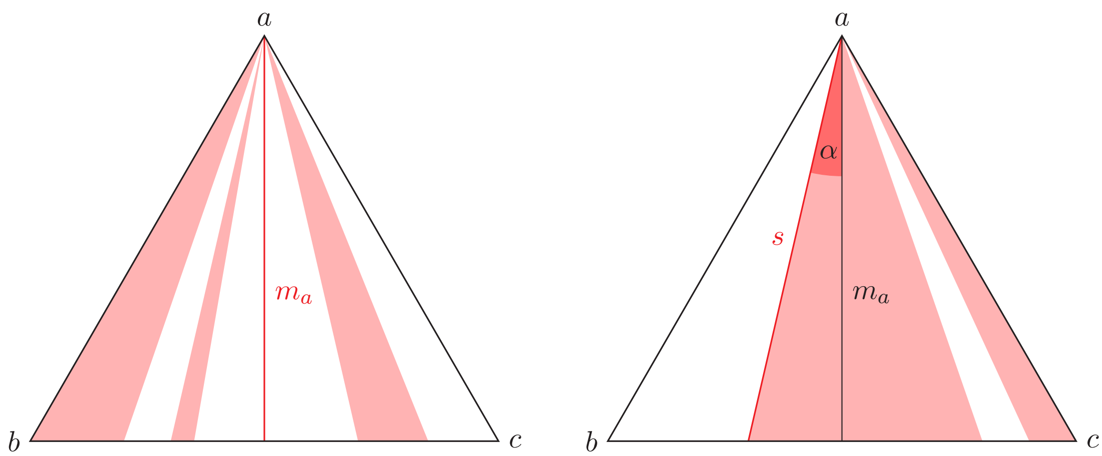


Figure 2.12. Worst case optimality: For $d \to 0$, this polygon requires a total floodlight angle of $(n-1)\frac{\pi}{6}$

3 Minimal Covering of Equilateral Triangles

In this chapter, we examine a minimal covering of equilateral triangles with vertex floodlights. An equilateral triangle is a triangle in which all three sides as well as the three internal angles are equal. The internal angles are $\frac{\pi}{3}$. Intuitively, a total floodlight angle of at least $\frac{\pi}{3}$ is necessary to cover an equilateral triangle completely by placing a floodlight on one vertex that covers the whole triangle. We present a formal proof of this intuition, whose key ideas are outlined in the following.

Consider an equilateral triangle and an arbitrary covering of it with some set of vertex floodlights. Let the vertices of the triangle be labeled as a, b and c. The angle bisector of a is the line passing through a and cuts its internal angle into two equal smaller angles. In the following, we write m_a for the part that is within the triangle. If the triangle is covered by floodlights on only one vertex, say a, we already require an angle of $\frac{\pi}{3}$ only to cover the side opposite to a. Assume the covering consists of floodlights on at least two vertices. In this case, the triangle is not covered completely by floodlights on vertex a. We have to consider two different cases (see Figure 3.1). If m_a is not covered by



(a) Case 1: m_a is not covered by a floodlight on a. As a result, an angle of $\frac{\pi}{3}$ is required to cover m_a with floodlights on b and c.

(b) Case 2: m_a is covered by a floodlight on a. We select a line segment s that is not covered by a and examine the angle that is required to cover it with floodlights on b and c.

Figure 3.1. The red filled areas are the visibility polygons of the floodlights placed on a.

a floodlight on vertex a, an angle of $\frac{\pi}{3}$ is required just to cover this line segment with floodlights on the vertices b and c. Otherwise, we can find another line segment s passing through a that is not covered by a floodlight on a. With a good choice of s, we obtain an angle $0 < \alpha < \frac{\pi}{6}$ and may therefore assume that a floodlight with an angle of at least $2(\alpha - \varepsilon)$ for an arbitrarily small $\varepsilon > 0$ is placed on a and covers m_a. We show that an angle of at least $\frac{\pi}{3} - 2\alpha$ is necessary just to cover the line segment s with floodlights on the vertices b and c. With the floodlight on a, whose angle is at least $2(\alpha - \varepsilon)$, follows a total floodlight angle of at least $2(\alpha - \varepsilon) + \frac{\pi}{3} - 2\alpha = \frac{\pi}{3} - 2\varepsilon$ for an arbitrarily small $\varepsilon > 0$.

Theorem 2. *Given an equilateral triangle, a total floodlight angle of at least* $\frac{\pi}{3}$ *is required to cover it completely.*

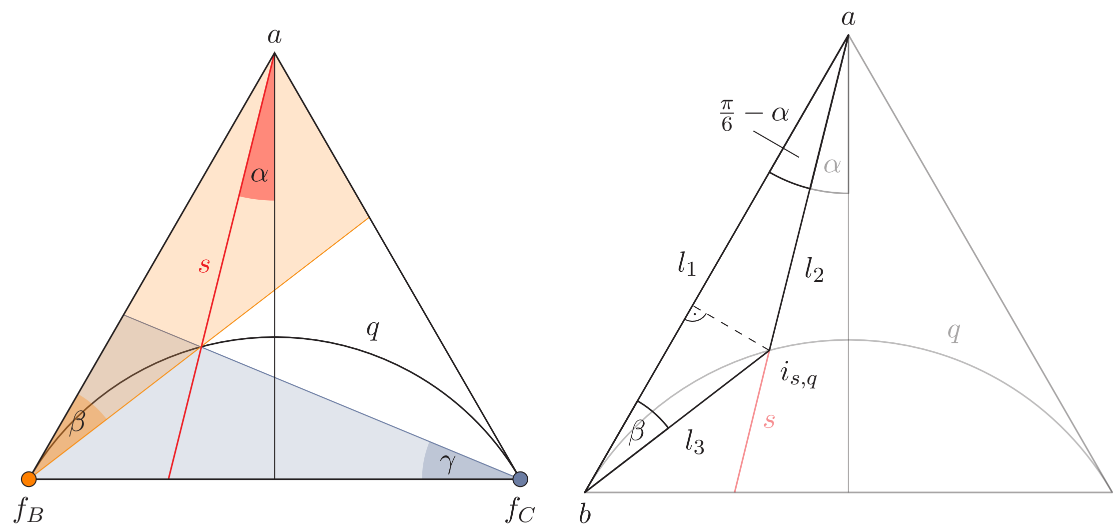
Proof. Without loss of generality, the triangle is has side lengths of 1 and is defined by the points $a = \left(0, \frac{\sqrt{3}}{2}\right)$, $b = (-\frac{1}{2}, 0)$ and $c = (\frac{1}{2}, 0)$ in the following. Consider an arbitrary feasible covering $\mathcal{C}$ of it. If floodlights are only placed on one vertex, say a, an angle of at least $\frac{\pi}{3}$ is required just to cover the side opposite to it. Otherwise, we distinguish between two different cases, as explained above.

Case 1: m_a is not covered by a floodlight on vertex a

Assume the angle bisector m_a is not covered by a floodlight on vertex a (see Figure 3.1a). Hence, s is completely covered by floodlights on b and c in $\mathcal{C}$. The covering of m_a consists only of floodlights on the vertices b and c. Using the fact that the distance between b and any point on m_a is the same as the distance between c and this point, every floodlight on c can be *flipped* along m_a. Formally, each floodlight (c, δ_1, δ_2) with $\frac{2\pi}{3} \leq \delta_1 < \pi$ and $\delta_1 < \delta_2 \leq \pi$ can be replaced by a floodlight $(b, \pi - \delta_2, \pi - \delta_1)$, which covers the same part of the perpendicular bisector with the same angle. As a result, a covering of s with floodlights on vertex b only can be achieved with the same total floodlight angle as the covering of s in $\mathcal{C}$. Since m_a is covered completely by floodlights on b, the angle has to be at least $\frac{\pi}{3}$.

Case 2: m_a is covered by a floodlight on vertex a

If the first case does not hold, a floodlight on vertex a covers m_a. In this case, we consider the boundaries of its visibility polygon. Without loss of generality, we may assume that the left boundary is closer to the bisector than the right one. Let α' be the angle between the left boundary and the bisector. We consider a line segment s passing through a that lies close to the left boundary and is not covered by a floodlight on a. Let the angle between s and m_a be $\alpha = \alpha' + \varepsilon$ (see Figure 3.1b). We show that the overall angle, required to cover s by floodlights on b and c, is at least $\frac{\pi}{3} - 2\alpha$. Since the floodlight, placed on a, has an angle of at least $\alpha - \varepsilon$ on each side of m_a, we obtain a total floodlight angle of at least $\frac{\pi}{3} - 2\varepsilon$ for an arbitrarily small $\varepsilon > 0$. To examine the total floodlight angle that is at least required to cover s with floodlights on b and c only, we make use of two lemmas. Their correctness is shown at the end of this proof.



(a) A minimum covering of s consists of two floodlights f_b and f_c with angles $\beta = \gamma$. The point on s, where the covering switches from c to b is on the arc q.

(b) The angle β is smaller than or equal to $\frac{\pi}{6} - \alpha$, since $l_3 \leq l_2$ for all $0 < \alpha < \frac{\pi}{6}$.

Figure 3.2. Visualization of a minimal covering of s.

Lemma 2. *Consider a covering $\mathcal{C}_s$ of s, only containing floodlights, placed on b or c. There are two angles $0 \leq \beta, \gamma \leq \frac{\pi}{3}$ with $\beta + \gamma \leq \measuredangle_{\mathcal{C}_s}$, such that two floodlights $f_b = (b, \frac{\pi}{3} - \beta, \frac{\pi}{3})$ and $f_c = (c, \pi - \gamma, \pi)$ are sufficient to cover s completely (see Figure 3.2a).*

Lemma 3. *A minimum covering $\mathcal{C}_s^{min}$ of s consists of two floodlights $f_b = (b, \frac{\pi}{3} - \beta, \frac{\pi}{3})$ and $f_c = (c, \pi - \gamma, \pi)$ with $\beta = \gamma$. Furthermore, the point on s, where the covering switches from c to b, is on the arc q defined by Equation 3.1 (see Figure 3.2a).*

$$q : x^2 + \left(y + \frac{1}{2\sqrt{3}}\right)^2 = \frac{1}{3}, x \geq 0 \tag{3.1}$$

The coverage of s in $\mathcal{C}$ is at most as good as an optimal covering of s. Since s is covered in $\mathcal{C}$ by floodlights placed only on b and c, we may conclude that this covering of s is at most as good as a covering with two floodlights f_b and f_c with angles β and γ and $\beta = \gamma$ as defined in Lemma 2. By the choice of s, a floodlight on vertex a covers an area with an angle of at least $(\alpha - \varepsilon)$ on both sides of m_a. Finally, it remains to show that $\beta + \gamma + 2(\alpha - \varepsilon) \geq \frac{\pi}{3}$. Consider the triangle, defined by a, b and $i_{s,q}$, where $i_{s,q}$ is the intersection point of s and c (see Figure 3.2b). The side l_1 between a and b is the longest side of the triangle for any $0 < \alpha < \frac{\pi}{6}$. Therefore, the angle β is greater than or equal to $\frac{\pi}{6} - \alpha$, if and only if the length of the side l_2 between a and $i_{s,q}$ is greater than or equal to the length of the side l_3 between b and $i_{s,q}$. Since $i_{s,q}$ is on the arc q, l_2 gets greater and l_3 gets smaller when increasing the size of α. As a result, it is sufficient to show that $l_2 \geq l_3$ for $\alpha = 0$.

For an angle $\alpha = 0$, the intersection of s and the arc c is at $x = 0$. Evaluating c at $x = 0$ yields

$$\begin{aligned} & x^2 + \left(y + \frac{1}{2\sqrt{3}}\right)^2 = \frac{1}{3} \\ \overset{x=0}{\iff} \quad & \pm\left(y + \frac{1}{2\sqrt{3}}\right) = \frac{1}{\sqrt{3}} \\ \iff \quad & y_1 = \frac{1}{2\sqrt{3}},\ y_2 = -\frac{\sqrt{3}}{2}. \end{aligned}$$

Only y_1 is within the triangle. This leads to lengths of l_2 and l_3 of

$$\begin{aligned} l_2 &= \sqrt{(0-0)^2 + \left(\frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}}\right)^2} = \frac{1}{\sqrt{3}} \\ l_3 &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{3}}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{12}} = \frac{1}{\sqrt{3}}. \end{aligned}$$

As suspected, $l_2 \geq l_3$ holds for all $0 < \alpha < \frac{\pi}{3}$, which leads to $\beta \geq \frac{\pi}{6} - \alpha$. With $\beta = \gamma$ we obtain

$$\beta + \gamma + 2(\alpha - \varepsilon) \overset{\beta=\gamma}{=} 2(\beta + \alpha - \varepsilon) \overset{\beta \geq \frac{\pi}{6} - \alpha}{\geq} 2\left(\frac{\pi}{6} - \alpha + \alpha - \varepsilon\right) = \frac{\pi}{3} - 2\varepsilon.$$

This angle is the at least required angle to cover the equilateral triangle completely for an arbitrary small $\varepsilon > 0$. Assume there is a feasible covering with a total floodlight angle of $\frac{\pi}{3} - x < \frac{\pi}{3}$. Selecting $\varepsilon = \frac{x}{2}$ leads to a contradiction. Hence, an angle of $\frac{\pi}{3}$ is required to cover an equilateral triangle completely. □

It remains to show the correctness of the two previously used lemmas.

Repetition of Lemma 2. Consider a covering $\mathcal{C}_s$ of s, only containing floodlights, placed on b or c. There are two angles $0 \leq \beta, \gamma \leq \frac{\pi}{3}$ with $\beta + \gamma \leq \measuredangle_{\mathcal{C}_f}$, such that two floodlights $f_b = \left(b, \frac{\pi}{3} - \beta, \frac{\pi}{3}\right)$ and $f_c = (c, \pi - \gamma, \pi)$ are sufficient to cover s completely.

Proof. Consider an arbitrary covering $\mathcal{C}_s = f_0, \ldots, f_m, m \in \mathbb{N}$ of the line segment s with floodlights placed only on the vertices b and c. Each floodlight $f_i, 0 \leq i \leq m$ covers a part of s, defined by an interval $I_i = [y_i, y_i']$ on the y axis with $0 \leq y_i < y_i' \leq \frac{\sqrt{3}}{2}$ (see Figure 3.3). The floodlights are ordered by increasing y_i values of the corresponding intervals. Formally, for two intervals I_k and I_l with $k < l$, holds $y_k < y_l$. We ignore floodlights that cover an interval on s that is already fully covered by one other floodlight, since these floodlights do not affect the covering of s. Hence, $y_l < y_k \iff y_l' < y_k'$ holds. Floodlights that cover an interval which is already covered by a set of floodlights, but not by one other floodlight alone, will not be ignored. Following, we examine for each such interval, whether a floodlight on b or a floodlight on c could cover it with a smaller floodlight angle. Let k be the minimal index, such that I_k could be covered by a floodlight placed on b more efficiently than by one placed on c, i.e. with a strictly smaller

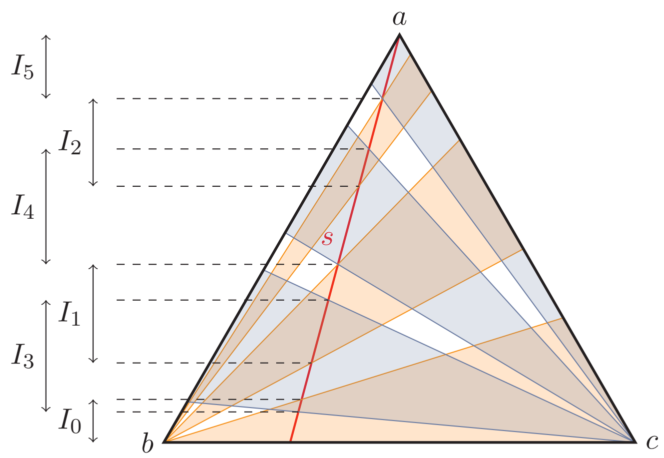


Figure 3.3. An exemplary covering of s with floodlights on b and c. The covering consists of six floodlights that cover six non-disjoint intervals $I_1, \dots, I_6$ on s.

floodlight angle. We show that for all intervals I_m with $m > k$, a floodlight on b would also be the more efficient choice. As a result, the total floodlight angle of $\mathcal{C}_s$ is greater than or equal to the total floodlight angle of a covering of s, where one floodlight, placed on c, covers the interval $\left[0, y'_{k-1}\right]$ and a second floodlight, placed on b, covers the interval $\left[y_k, \frac{\sqrt{3}}{2}\right]$.

The proof is split in two parts. Initially, we show that an increase of y'_k does not affect the inequality $\beta < \gamma$, meaning the interval $[y_k, y']$ can also be covered more efficiently by a floodlight placed on b than by a floodlight placed on c for any $\frac{\sqrt{3}}{2} \geq y' > y'_k$. Secondly, we prove that an increase of y_k does not affect the inequality $\beta < \gamma$, either. Formally, we show

(1) $\beta(y_k, y'_k) < \gamma(y_k, y'_k) \Rightarrow \beta(y_k, y') < \gamma(y_k, y')$ for all $y_k < y'_k < y' \leq \frac{\sqrt{3}}{2}$,

(2) $\beta(y_k, y') < \gamma(y_k, y') \Rightarrow \beta(y, y') < \gamma(y, y')$ for all $y_k < y < y'$,

where $\beta(y, y')$ and $\gamma(y, y')$ describes the required angles to cover the interval $[y, y']$ by floodlights placed on b and c, respectively.

Since the conclusion of (1) is the premise of (2), the successive execution of both increasing steps allows an adaption to all intervals I_i with $i > k$.

We determine the functions β_0 and γ_0, mapping an y coordinate to the angle that is required to cover the interval $[0, y]$. Utilizing these functions, we can propose the functions

$$\begin{aligned}\beta(y, y') &= \beta_0(y') - \beta_0(y) \text{ and}\\ \gamma(y, y') &= \gamma_0(y') - \gamma_0(y).\end{aligned}$$

By applying the trigonometric functions, we obtain the following functions for β_0 and γ_0:

$$\beta_0(y) = \arctan\left(\frac{y}{\frac{1}{2} - \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)}\right)$$
$$\gamma_0(y) = \arctan\left(\frac{y}{\frac{1}{2} + \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)}\right).$$

The following characteristics of β_0 and γ_0 are proven in Appendix B.2:

(1) $\beta_0(y) \geq \gamma_0(y)$ for all $0 \leq y \leq \frac{\sqrt{3}}{2}$

(2) The curves of the derivatives $\frac{d}{dx}\beta_0$ and $\frac{d}{dx}\gamma_0$ have a unique intersection within the relevant domain $\left[0, \frac{\sqrt{3}}{2}\right]$

(3) $\frac{d}{dx}\gamma_0 < \frac{d}{dx}\beta_0$ at $y = 0$

(4) $\frac{d}{dx}\beta_0 < \frac{d}{dx}\gamma_0$ at $y = \frac{\sqrt{3}}{2}$

At first, we examine an increase of the upper interval boundary y'_k. Since the angle $\beta(y_k, y'_k)$ is smaller than $\gamma(y_k, y'_k)$, it follows with (1), that the gradient of β_0 is smaller than the gradient of γ_0 in at least one point between 0 and y'_k. With (2), (3) and (4) we can conclude that the gradient of β_0 is smaller than the gradient of γ_0 for all $y'_k < y < \frac{\sqrt{3}}{2}$. As a result, the function $\beta(y_k, y') = \beta_0(y') - \beta_0(y_k)$ is also smaller than $\gamma(y_k, y') = \gamma_0(y') - \gamma_0(y_k)$.

Next, we consider an increase of the lower interval boundary y_k. If the gradient of β_0 is smaller than the gradient of γ_0 at y_k, it follows that it is smaller than γ_0 in the whole interval $[y_k, y']$, especially in the interval $[y, y']$, too. Hence, $\beta(y, y') \leq \gamma(y, y')$. Otherwise, the gradient of β_0 is greater or equal to the gradient of γ_0 at y_k. In this case, we can increase y_k until the gradient of β_0 is smaller without the inequality between β and γ being affected. This constellation is equivalent to the first case, where the gradient of β_0 is smaller than the gradient of γ_0 for the whole remaining interval. □

Repetition of Lemma 3. A minimum covering $\mathcal{C}_s^{\min}$ of s consists of two floodlights $f_b = \left(b, \frac{\pi}{3} - \beta, \frac{\pi}{3}\right)$ and $f_c = (c, \pi - \gamma, \pi)$ with $\beta = \gamma$. Furthermore, the point on s, where the covering switches from c to b, is on the arc q defined by Equation 3.1.

Proof. With Lemma 2 it is proven, that a minimal covering of s consists of two floodlights, one on vertex b that covers the upper part of s and one on vertex c that covers the lower part.

We define the functions $\beta : \left(0, \frac{\sqrt{3}}{2}\right) \rightarrow \left(0, \frac{\pi}{3}\right)$ and $\gamma : \left(0, \frac{\sqrt{3}}{2}\right) \rightarrow \left(0, \frac{\pi}{3}\right)$, which describe the angles β and γ with respect to the y coordinate at which the covering switches from c to b. These functions can be determined by applying the trigonometric functions (see Figure 3.4) and are given by

$$\beta(y) = \arctan\left(\frac{l_1}{y}\right) - \frac{\pi}{6} = \arctan\left(\frac{\frac{1}{2} - l_3}{y}\right) - \frac{\pi}{6}$$
$$= \arctan\left(\frac{\frac{1}{2} - \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)}{y}\right) - \frac{\pi}{6} \tag{3.2}$$

$$\gamma(y) = \arctan\left(\frac{y}{l_2}\right) = \arctan\left(\frac{y}{\frac{1}{2} + l_3}\right) = \arctan\left(\frac{y}{\frac{1}{2} + \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)}\right) \tag{3.3}$$

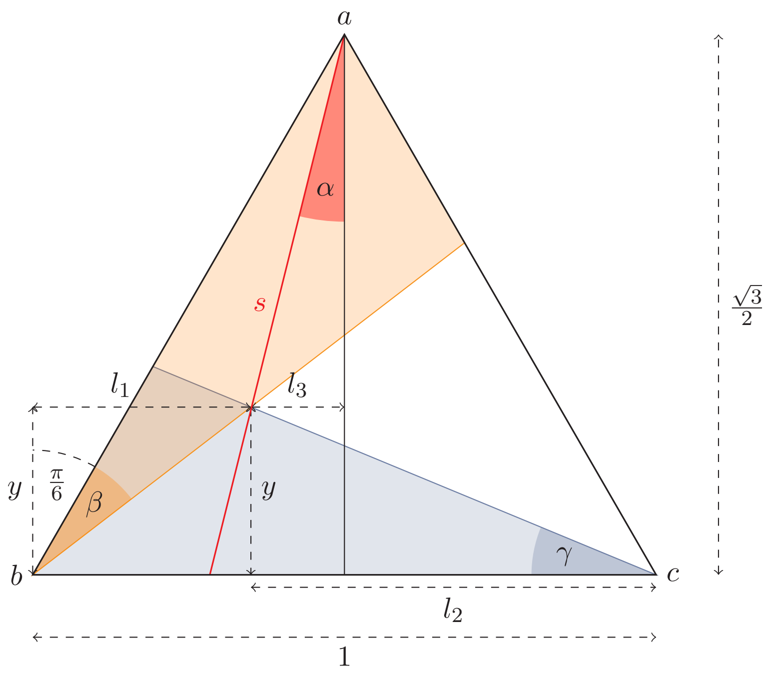


Figure 3.4. The angles β and γ depend on y and can be determined by applying the trigonometric functions.

The function $\beta + \gamma(y)$, obtained by summing over both functions, has a global minimum in the domain $\left(0, \frac{\sqrt{3}}{2}\right)$. We will show that this minimum is reached, if and only if $\beta(y) = \gamma(y)$ holds. This is visualized in the plot in Figure 3.5.

The proof is split into two steps. The first step is to show, that the global minimum is at the y coordinate of the intersection of s and q, labeled as $i^y_{s,q}$. The extreme value determination and the determination of the equality to the intersection point is given in Appendix B.3 and B.4. The second step is to show that $\beta(i^y_{s,q}) = \gamma(i^y_{s,q})$ holds. For this, consider Figure 3.6. According to the Inscribed Angle Theorem (see Appendix B.1), the angle δ stays the same regardless of α. For $\alpha = 0$ we obtain

$$\widetilde{\beta} = \gamma = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

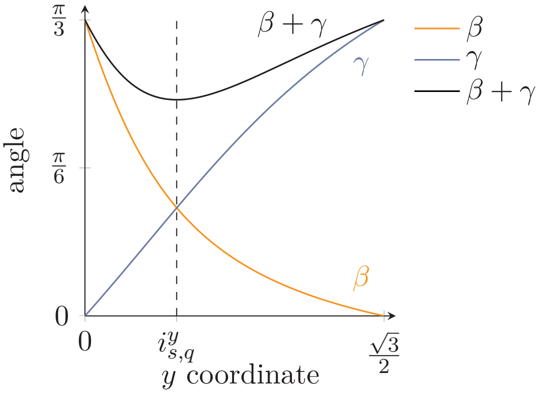


Figure 3.5. The curves of β, γ and the sum $\beta + \gamma$ for $\alpha = \frac{\pi}{12}$. The sum $\beta + \gamma$ reaches a global minimum, when $\beta = \gamma$ holds. This is at the intersection $i^y_{s,q}$ of s and q.

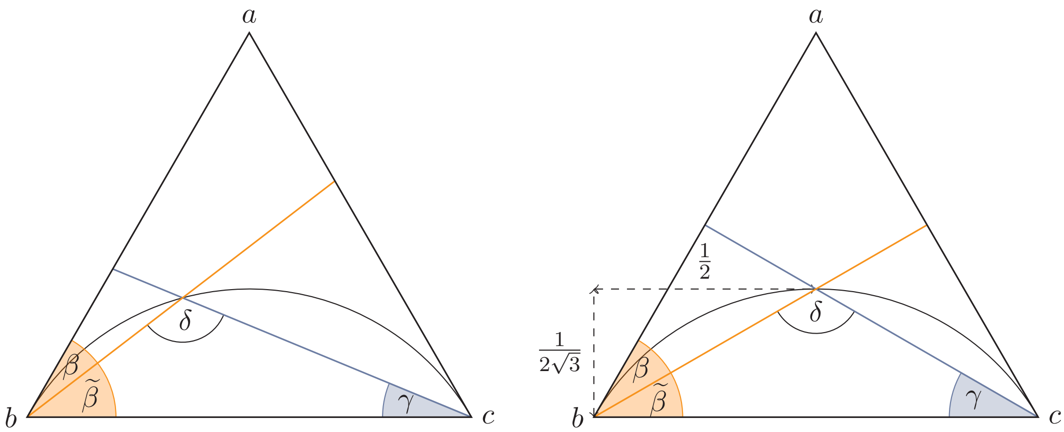


Figure 3.6. According to the Inscribed Angle Theorem, $\beta = \gamma$ holds at the intersection point of s and q

With $\widetilde{\beta} + \gamma + \delta = \pi$ we get $\delta = \frac{2\pi}{3}$, which holds for every $\alpha \in \left(0, \frac{\pi}{6}\right)$. As a result, we can describe $\widetilde{\beta}$ with respect to γ for all α as $\widetilde{\beta} = \pi - \frac{2\pi}{3} - \gamma = \frac{\pi}{3} - \gamma$. With $\widetilde{\beta} + \beta = \frac{\pi}{3}$ follows

$$\beta = \frac{\pi}{3} - \widetilde{\beta} = \frac{\pi}{3} - \left(\frac{\pi}{3} - \gamma\right) = \gamma.$$

□

4 Upper Bound for Simple Polygons

In this chapter, we derive a universal upper bound of $(n-2)\frac{\pi}{4}$ for simple polygons with n vertices. We give a formal proof, which includes an efficient algorithm to determine a covering, satisfying this upper bound. The idea of the proof is to partition a given polygon in subpolygons and cover these subpolygons independently, similar to Chvátal's proof of the Art Gallery Theorem[6]. Furthermore, there similarities to Fisk's alternate proof, since the dual graph of a determined triangulation is used to find the subpolygons [16].

Initially, we give some definitions about elementary graph theory, used in this chapter. A tree $T = (V, E)$ is a connected graph with vertices V and edges $E \subseteq V \times V$ that does not contain any cycles, meaning any two vertices are connected by exactly one path. In the following, we consider only undirected trees. The distance between two vertices $v, w \in V$ is the number of edges on the path between v and w. The degree of a vertex is the number of its incident edges and the maximum degree of a tree T is written as $\Delta(T)$.

Lemma 4. *Each undirected tree $T = (V, E)$ with $|V| > 1$ and $\Delta(T) \leq 3$ contains at least one of the two structures given in Figure 4.1.*

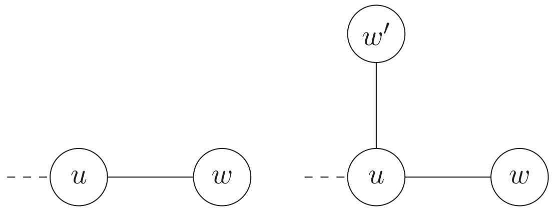


Figure 4.1. At least one of these structures exists in every tree with $|V| > 1$ and $\Delta(T) \leq 3$.

Proof. Consider any vertex $v \in V$. Let w be the vertex with a maximum distance to v. w is a leaf and its predecessor u has at most one other neighbor w' besides the vertex on the path to v, since the maximum degree is 3. w' is also a leaf, since another neighbor of w' would have a larger distance to v as w, which is contradiction. □

Lemma 5. *Let $T = (V, E)$ be a tree with $|V| > 1$ and $\Delta(G) \leq 3$.*
There is a partitioning of V in subsets $V_1 \dot\cup \ldots \dot\cup V_k = V$ with $2 \leq |V_i| \leq 3$ for all $0 < i \leq k-1$ and $1 \leq |V_k| \leq 3$, such that the vertices in each subset are adjacent in G. If $|V_k| = 1$, there is at least one subset of size 3.

Proof. By Lemma 4, there is at least one of the structures described above in T. Removing an arbitrary representant to obtain T' still yields a tree and Lemma 4 can be applied recursively until a tree with $|V| \leq 4$ has been obtained. Since we remove at most three vertices in each step, the remaining graph contains at least two vertices. If $|V| \in \{2,3\}$, V forms the last partition. Otherwise, $|V| = 4$ and the remaining graph is structured either as a chain or as a star (see Figure 4.2). In the first case, the four vertices can be split into two partitions of size 2. In the second case, we obtain one partition of size 3 and one of size 1. $\square$

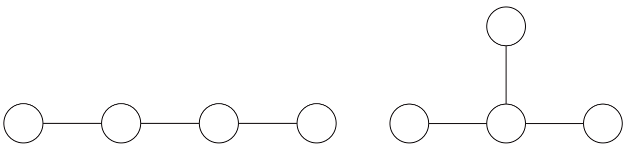


Figure 4.2. The only two trees with $|V| = 4$ and $\Delta(G) \leq 3$.

Lemma 6. *Given a polygon $\mathcal{P}$ with $n \in \{3,4,5\}$ vertices, there is a feasible covering $\mathcal{C}$ with*

$$\measuredangle_{\mathcal{C}} \leq (n-1)\frac{\pi}{6}.$$

Proof. Consider a polygon $\mathcal{P}$ with $n \in \{3,4,5\}$ vertices.

$n = 3$: $\mathcal{P}$ is a triangle and can be covered with a guard placed on the vertex with the smallest internal angle, which is $\leq \frac{\pi}{3}$.

$n = 4$: If $\mathcal{P}$ is convex, a guard on the vertex with the smallest internal angle, which is at most $\frac{\pi}{2}$, covers it completely. Otherwise $\mathcal{P}$ has one reflex vertex v_r. If the angle of the vertex v_r' opposite to v_r is $v_r' \leq \frac{\pi}{2}$, a guard on it covers $\mathcal{P}$ completely. Otherwise, the remaining two angles ϕ_1 and ϕ_2 have to satisfy $\phi_1 + \phi_2 < \frac{\pi}{2}$, since the sum of all internal angles is 2π and the angle of the reflex vertex is $v_r > \pi$. Two guards on these vertices cover $\mathcal{P}$ completely (see Figure 4.3).

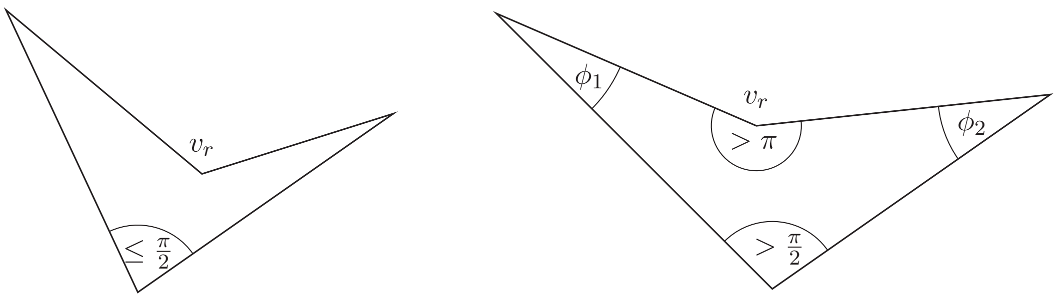


Figure 4.3. In a reflex quadrangle, either the vertex opposite to the reflex one is or the sum $\phi_1 + \phi_2$ is $\leq \frac{\pi}{2}$.

$n = 5$**:** A polygon with 5 vertices is called pentagon. We differentiate the possible occurring constellations with respect to the number and order of reflex vertices. A pentagon has at most two reflex vertices, lying side by side or with one convex vertex lying in between. In the following, the order of convex and reflex vertices is described as a chain of letters, where c is a convex and r a reflex vertex. $\mathcal{P}$ is labeled with vertices a, b, c, d and e and internal angles $\phi_a, \phi_b, \phi_c, \phi_d$ and ϕ_e. If a vertex $x \in \{a, b, c, d, e\}$ is reflex, the angle ϕ'_x is defined as $\phi'_x = \phi_x - \pi$, describing only the overlapping part of the internal angle $\phi_x > \pi$.

Case 1 (c - c - c - c - c): $\mathcal{P}$ is convex. A floodlight on the vertex with the smallest internal angle, which is at most $\frac{3\pi}{5} < 4\frac{\pi}{6}$, covers $\mathcal{P}$ completely.

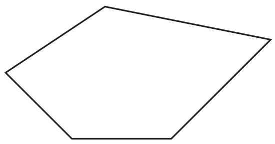
Case 2 (r - c - c - c - c): One vertex of the pentagon, e.g. a, is reflex. We differentiate between two cases.

Case 2.1: The polygon can be partitioned into two triangles (see Figure 4.4b). Each triangle can be covered with an angle of at most $\frac{\pi}{3}$. When a guard is placed on an edge, but not on a vertex, it can be split and shifted to two vertices without increasing its angle or restricting its visibility polygon (see Figure 4.4c).

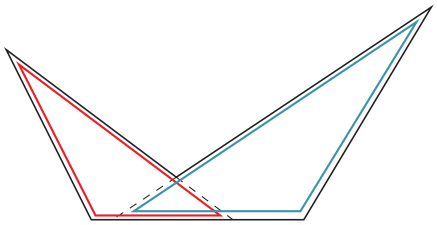
Case 2.2: The whole polygon is visible from at most one of the two vertices not adjacent to a, say d (see Figure 4.4d). Since the sum of the internal angles of a polygon with n vertices is $(n-2)\pi$, 3π is the internal angle sum in a pentagon. We obtain $\phi'_a + \phi_b + \phi_c + \phi_d + \phi_e = 2\pi$. If $\phi_d \leq \frac{2\pi}{3}$, a floodlight on d covers $\mathcal{P}$. Assume $\phi_d > \frac{2\pi}{3}$. We obtain $\phi'_a + \phi_b + \phi_c + \phi_e < \frac{4\pi}{3}$. Two floodlights placed either on a and c or b and e with respective covering angles of $\phi'_a + \phi_c$ and $\phi_b + \phi_e$ would suffice to cover $\mathcal{P}$ completely. Either $\phi'_a + \phi_c \leq \frac{2\pi}{3}$ or $\phi_b + \phi_e \leq \frac{2\pi}{3}$.

Case 3 (r - r - c - c - c): Two reflex vertices, i.e. a and b, are adjacent in $\mathcal{P}$, as visualized in Figure 4.4e. Without the reflex parts of the reflex vertices, an angle of $\phi'_a + \phi'_b + \phi_c + \phi_d + \phi_e = \pi$ remains. The whole polygon is visible from d, which is not adjacent to a and b. Hence, a guard on d covers $\mathcal{P}$, if $\phi_d \leq \frac{2\pi}{3}$. Assume that $\phi_d > \frac{2\pi}{3}$. The remaining angles amount to $\phi'_a + \phi'_b + \phi_c + \phi_e < \frac{\pi}{3}$ remains. Guards on a, c and e, oriented as the angles ϕ'_a, ϕ_c and ϕ_e, cover $\mathcal{P}$ completely.

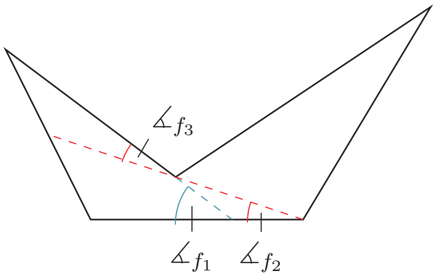
Case 4 (r - c - r - c - c): Consider a pentagon with two non-adjacent reflex vertices, i.g. a and c (see Figure 4.4f). Consider the only triangulation of $\mathcal{P}$, splitting $\mathcal{P}$ into three triangles abc, acd and ade. From d, two triangles are completely visible, the third one is visible from b. Furthermore, a floodlight on a and one on e with angles ϕ'_a and ϕ_e may cover all three triangles. Since $\phi'_a + \phi_b + \phi'_c + \phi_d + \phi_e = \pi$, either $\phi_d + \phi_b \leq \frac{\pi}{2}$ or $\phi'_a + \phi_e \leq \frac{\pi}{2}$. □



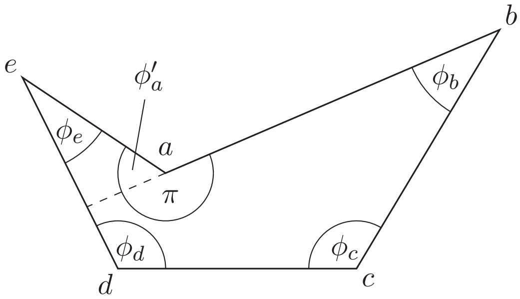
(a) Case 1: The pentagon is convex and can be covered by a guard with an angle $\leq \frac{3\pi}{5}$.



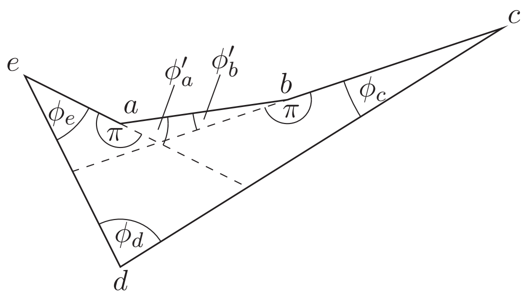
(b) Case 2.1: The pentagon can be split in two triangles, which can be covered each with an angle $\leq \frac{\pi}{3}$.



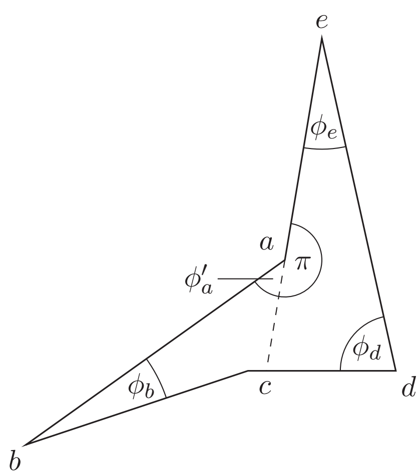
(c) A floodlight that is not placed on an edge can be replaced by vertex floodlights, cover the same or a larger area with an equal total covering angle.



(d) Case 2.2: $\mathcal{P}$ is completely visible from d, from a and c or from b and e. Either ϕ_d, $\phi'_a + \phi_c$ or $\phi_b + \phi_e$ is $\leq \frac{2\pi}{3}$, since the sum of all is 2π.



(e) Case 3: When $\phi_d > \frac{2\pi}{3}$, $\phi'_a + \phi_c + \phi_e < \frac{\pi}{3}$. Both cover $\mathcal{P}$ completely.



(f) Case 4: $\mathcal{P}$ can be covered by guards with angles ϕ'_a and ϕ_e or with angles ϕ_d and ϕ_b. Since an internal angle of π is split on $\phi'_a, \phi_b, \phi'_c, \phi_d$ and ϕ_e, one of these sums is at most $\frac{\pi}{2}$.

Figure 4.4. The five possible structures of a pentagon with respect to the number and order of reflex vertices.

Theorem 3 (Upper bound for simple polygons)**.** *Given a simple polygon $\mathcal{P}$, an angle of*

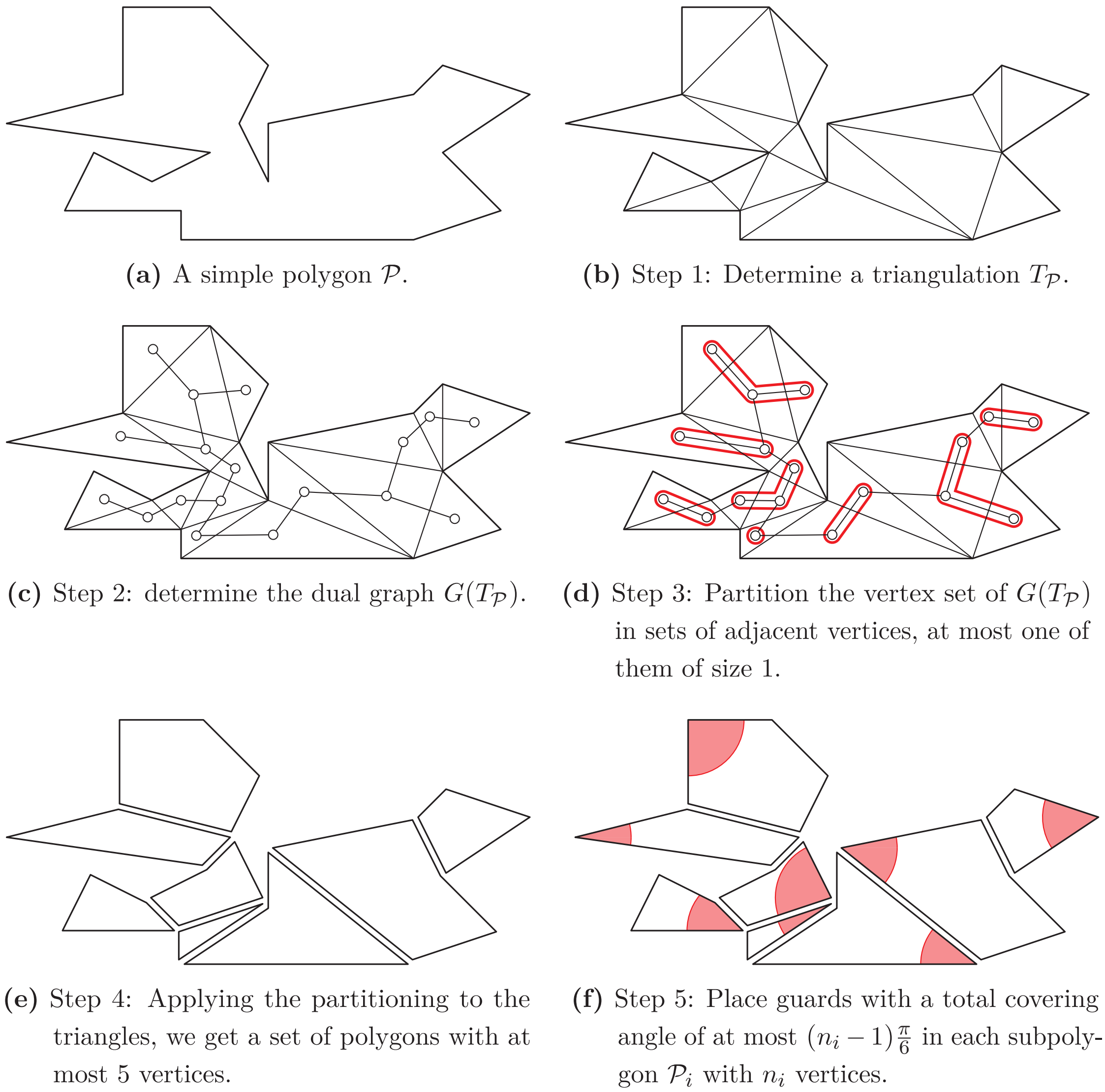
$$(n-2)\frac{\pi}{4}$$

is always sufficient, to cover $\mathcal{P}$ completely.

Proof. Each polygon can be triangulated into $n-2$ triangles. Given such a triangulation $T_{\mathcal{P}}$ of $\mathcal{P}$, we consider the dual graph $G(T_P) = (V, E)$ of the triangulation. V contains a vertex per triangle in T_P and two vertices are connected by an edge, if and only if the corresponding triangles share a (common) face. Since a triangle has only three faces, $G(T_{\mathcal{P}})$ has a maximum degree of $\Delta(G) \leq 3$. Furthermore, $G(T_{\mathcal{P}})$ is a tree. Using Lemma 5, the triangulation can be partitioned in at most $\frac{n-2}{2}$ sets of adjacent triangles of size ≤ 3. Each set forms a polygon with ≤ 5 vertices. Using Lemma 6, such a set of three triangles can be covered with an angle of at most $\frac{2\pi}{3} < 3\frac{\pi}{4}$ and a set of two triangles can be covered with an angle of at most $\frac{\pi}{2} = 2\frac{\pi}{4}$. A set of one triangle may occur at most once, but only together with at least one set of three triangles. These two sets can be covered with a total floodlight angle of at most $\frac{\pi}{3} + \frac{2\pi}{3} = 4\frac{\pi}{4}$. Altogether we obtain a covering angle of at most $\frac{\pi}{4}$ for each of the $n-2$ triangles, whose union forms the whole polygon $\mathcal{P}$. □

Lemma 7. *Given a simple polygon $\mathcal{P}$, a feasible covering $\mathcal{C}$ with $\measuredangle_{\mathcal{C}} \leq (n-2)\frac{\pi}{4}$ can be determined in $\mathcal{O}(n^2)$.*

Proof. A triangulation $T_{\mathcal{P}}$ can be determined in a simple way by using the *ear clipping method*: [24] explains that each polygon with $n > 3$ contains at least two *ears*, that being triangles where only one face is inside the polygon and the other two faces are edges of the polygon. In [11], an algorithm capable of finding such an ear in $\mathcal{O}(n)$ time is presented that finds. We may obtain a triangulation in $\mathcal{O}(n^2)$ by iteratively removing such an ear, until we are left with only a triangle. More complex algorithms are able to determine a triangulation in linear[5], respectively linear expected [2] time. After determining a triangulation $T_{\mathcal{P}}$, we can use the approach of the constructive proof of Lemma 4 to determine sets of adjacent vertices in $G(T_{\mathcal{P}})$ in $\mathcal{O}(n)$ for each set. Finally, each set of triangles is covered by placing floodlights in constant time as shown in Lemma 6. Overall, we get a time complexity of $\mathcal{O}(n^2)$. □



(a) A simple polygon $\mathcal{P}$.

(b) Step 1: Determine a triangulation $T_{\mathcal{P}}$.

(c) Step 2: determine the dual graph $G(T_{\mathcal{P}})$.

(d) Step 3: Partition the vertex set of $G(T_{\mathcal{P}})$ in sets of adjacent vertices, at most one of them of size 1.

(e) Step 4: Applying the partitioning to the triangles, we get a set of polygons with at most 5 vertices.

(f) Step 5: Place guards with a total covering angle of at most $(n_i - 1)\frac{\pi}{6}$ in each subpolygon $\mathcal{P}_i$ with n_i vertices.

Figure 4.5. Functioning of the algorithm to determine an $(n-2)\frac{\pi}{4}$ upper bound covering.

5 Further Aspects and Future Work

5.1. Improvement of the Upper Bound

The upper bound for simple polygons, given in Chapter 4, may be improved by assuming that an upper bound of $(n-1)\frac{\pi}{6}$ holds for simple polygons with $n < 2k+2$ vertices and a fixed constant $k \in \mathbb{N}$. As a result, a polygon could be partitioned in larger subpolygons, which leads to a smaller average covering angle per triangle in the triangulation.

Conjecture 1. *Given a simple polygon $\mathcal{P}$ with $n < 2k+2$ vertices, there is a covering $\mathcal{C}$, feasible for $\mathcal{P}$, with*

$$\measuredangle_{\mathcal{C}} \leq (n-1)\frac{\pi}{6}.$$

Given a graph $G = (V, E)$ and a vertex set $V' \subseteq V$, we define $G - V'$ as the graph G without the vertices in V' and their incident edges. The following lemma is a generalization of Lemma 4 and is used in Theorem 4 below.

Lemma 8. *Consider an undirected tree $T = (V, E)$ with $|V| \geq k$ vertices and $\Delta(T) \leq 3$. There is a vertex set $V' \subseteq V$ with $k \leq |V'| < 2k$, such that $T - V'$ stays connected.*

Proof. For $k \leq |V| < 2k$, the lemma holds with $V' = V$. Assume $|V| \geq 2k$. Consider the smallest subset $V' \subseteq V$ with $|V'| \geq k$, such that $T - V'$ is not empty and stays connected. Such a V' must always exists, since selecting all vertices except one leaf provides a valid solution. Consider the vertex $v \in V'$ that connects V' with the rest of T. If v has a degree of 2, there are at most $k-1$ other vertices in V', because otherwise $V' \setminus \{v\}$ would provide a smaller subset (see Figure 5.1, left). If the degree of v is 3, V' consists of v and two vertex subsets V'_1 and V'_2, connected by v. Each subset has to contain at most $k-1$ vertices, since otherwise, one of those would provide a smaller subset (see Figure 5.1, right). As a result, $|V'| = |V'_1| + |V'_2| + 1 \leq 2(k-1) + 1 = 2k - 1$. □

Assuming the correctness of Conjecture 1 for a fixed $k \in \mathbb{N}$, we can determine the following upper bound for simple polygons.

Theorem 4. *Assume Conjecture 1 holds for a fixed $k \in \mathbb{N}$. Given a simple polygon $\mathcal{P}$ with n vertices, a total floodlight angle of at most*

$$(n-2)\left(1+\frac{1}{k}\right)\frac{\pi}{6}$$

is always sufficient to cover $\mathcal{P}$ completely with vertex floodlights.

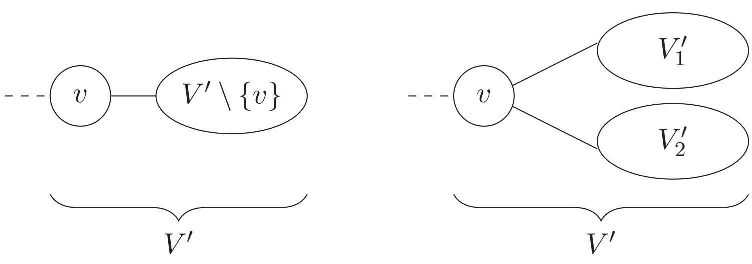


Figure 5.1. Let V' be the smallest subset, such that $|V'| \geq k$ and $G - V'$ stays connected. v is the vertex that connects V' with the rest of G. Since $\Delta(G) \leq 3$, V' takes one of these two forms. None of the subsets $V' \setminus \{v\}$, V'_1 and V'_2 has more than $k-1$ vertices, resulting in a size of $k \leq |V'| < 2k$.

Proof. Under the assumption of the correctness of Conjecture 1, this upper bound holds with the same arguments as the proven upper bound in Chapter 4. Consider a given polygon $\mathcal{P}$ with n vertices. If $n < 2k + 2$, we obtain a total floodlight angle of at most $(n-1)\frac{\pi}{6}$, according to Conjecture 1. Assume $n \geq 2k + 2$. Considering a triangulation $T_\mathcal{P}$ of $\mathcal{P}$, the dual graph $G(T_\mathcal{P})$ is a tree and has a maximum degree of $\Delta(G) \leq 3$. According to Lemma 8, $G(T_\mathcal{P})$ can be partitioned in subsets $V_1, \dots, V_{l-1}, l \leq n$ of adjacent vertices with sizes $k \leq |V_i| < 2k, 0 < i < l$, until the remaining graph contains $x_1 < 2k$ vertices, which forms the last subset V_l. The second last subset V_{l-1} contains at least $x_2 \geq 2k - x_1$ vertices, since otherwise V_l and V_{l-1} would form only one subset with less than $2k$ vertices.

Each of the subsets $V_i, 0 < i \leq l$, defines a polygon $\mathcal{P}_i$ with $|V_i| + 2$ vertices, formed by $|V_i|$ triangles in the triangulation. Using Conjecture 1, the first $l-2$ polygons $\mathcal{P}_1, \dots, \mathcal{P}_{l-2}$ can be covered with a total floodlight angle of at most $(|V_i| + 1)\frac{\pi}{6}$ each, resulting in an average angle per triangle of at most

$$\frac{|V_i| + 1}{|V_i|}\frac{\pi}{6} = \left(1 + \frac{1}{|V_i|}\right)\frac{\pi}{6} \overset{|V_i| \geq k}{\leq} \left(1 + \frac{1}{k}\right)\frac{\pi}{6}.$$

Consider the remaining two subpolygons $\mathcal{P}_{l-1}$ and $\mathcal{P}_l$ with $x_1 + 2$ and $x_2 + 2 \geq 2k - x_1 + 2$ vertices, respectively. The total floodlight angle for these subpolygons is at most $(x_1 + 1)\frac{\pi}{6}$ and $(x_2 + 1)\frac{\pi}{6}$, respectively. We obtain an overall angle for the last two subpolygons with $x_1 + x_2$ triangles in the triangulations of at most

$$(x_1 + 1)\frac{\pi}{6} + (x_2 + 1)\frac{\pi}{6} = (x_1 + x_2 + 2)\frac{\pi}{6}.$$

This leads to an average angle per triangle of

$$\frac{x_1 + x_2 + 2}{x_1 + x_2}\frac{\pi}{6} = \left(1 + \frac{2}{x_1 + x_2}\right)\frac{\pi}{6} \overset{x_2 \geq 2k - x_1}{\leq} \left(1 + \frac{2}{x_1 + 2k - x_1}\right)\frac{\pi}{6} = \left(1 + \frac{1}{k}\right)\frac{\pi}{6}. \quad \square$$

With this result, the gap between the assumed tight upper bound of $(n-1)\frac{\pi}{6}$ and a proven upper bound can be reduced by proving the tight upper bound for a set of polygons, restricted by their size.

5.2. Duality to Independent Circle Packing

Given a polygon $\mathcal{P}$, two set of points $P_1, P_2 \subseteq \mathcal{P}$ are called *independent* in $\mathcal{P}$, if and only if there are no two points $p \in P_1$ and $q \in P_2$, such that the line segment $\overline{pq}$ is contained in $\mathcal{P}$.

Consider the following problem of independent circle packing (ICP): Given a polygon $\mathcal{P}$, find a set of independent circles placed within $\mathcal{P}$, such that the minimum total angle, required to cover them with floodlights, is as large as possible (see Figure 5.2). We will examine the duality between this maximization problem and the AAGP. Ones can see easily that an optimal ICP solution always provides a lower bound to the AAGP. Since the circles are independent, each of them is covered independently in the AAGP, too. As a result, we obtain a duality between the minimization problem AAGP and the maximization problem ICP.

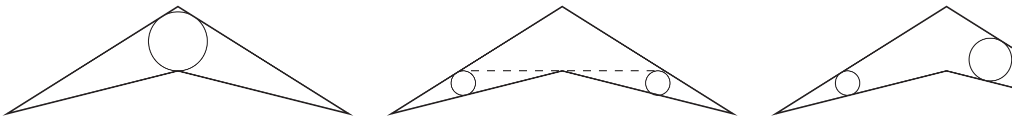


Figure 5.2. The both circle packings to the left are independent. The packing with one circle is probably better than the one with two vertices, since the total floodlight angle, required to cover the large circle is probably larger than the one, required to cover the two smaller circles. The right one is no valid packing, since the two circles are not independent.

To explore a duality gap between these problems, we assume the correctness of the following conjecture.

Conjecture 2. *Given two circles c_r and c_R with the same center and radii $r > 0$ and $R > r$, a minimum total floodlight angle of at least $2 \arcsin\left(\frac{r}{R}\right)$ is required to cover c_r with floodlights placed anywhere within c_R (see Figure 5.3).*

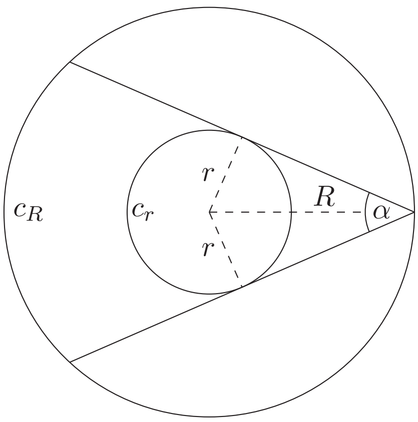


Figure 5.3. Assuming the correctness of Conjecture 2, a total floodlight angle of $\alpha = 2 \arcsin\left(\frac{r}{R}\right)$ is at least required to cover c_r with floodlights placed within c_R.

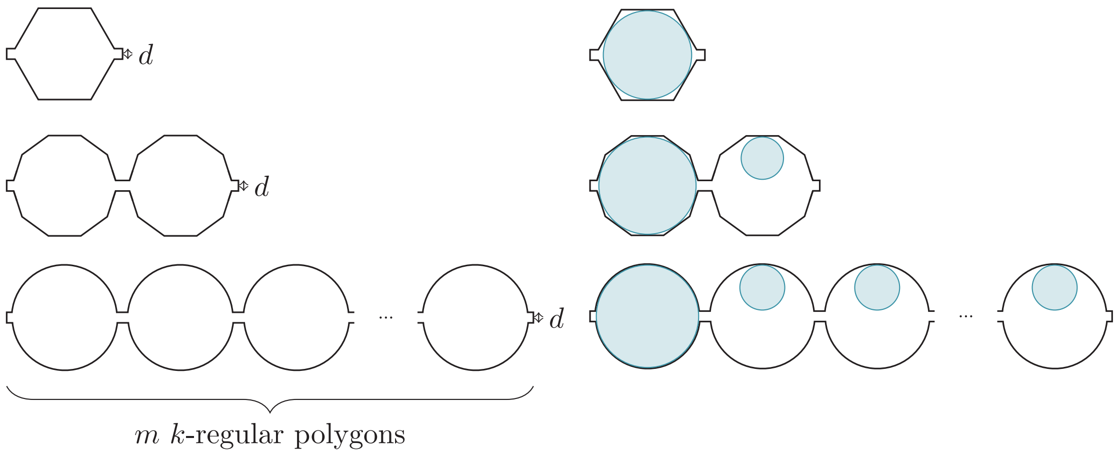


Figure 5.4. By connecting regular polygons with a small well, only one of these polygons can be fully packed with a circle. The well prevents from packing circles of size larger than half the inner polygon diameter in the other regular poolygons. In contrast, each of these subpolygons has to be covered nearly independently in the AAGP.

Consider the polygon structure given in Figure 5.4. The polygon is constructed from m regular polygons with k edges each, connected by a small well of height d. By decreasing $d \to 0$, each of the subpolygons has to be covered almost independently in the AAGP. Furthermore, by increasing $k \to \infty$, the regular polygons approach circles. We obtain the following packing in the ICP: A circle, spanned over the whole inner diameter of a regular polygon, may only occur once. The remaining $m-1$ regular polygons are maximal packed with one circle of size close the half of the inner diameter of the regular polygon. The well through the middle of each subpolygon prevents of packing a larger circle, which would result in a larger required covering angle. We obtain the following ICP solution, derived from Figure 5.5 assuming the correctness of Conjecture 2:

$$\mathrm{ICP}(\mathcal{P}) = \overbrace{(k-2)\frac{\pi}{k}}^{\text{one large circle}} + \overbrace{2(m-1)\arcsin\left(\frac{1}{3}\right)}^{m-1\text{ circles of half size}}$$

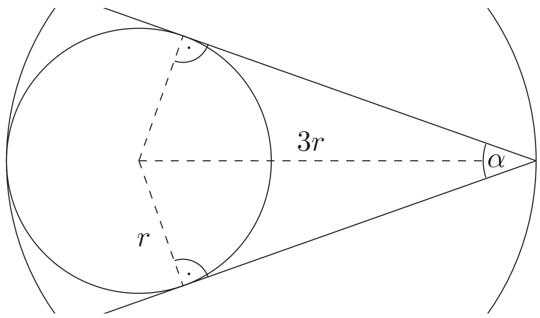


Figure 5.5. One of the smaller circles may be covered by an angle of $\alpha = 2\arcsin\left(\frac{1}{3}\right)$.

In contrast, each of the circles has to be covered nearly independently in the AAGP. Using Conjecture 2, we obtain a minimum total covering angle of almost

$$\mathrm{AAGP}(\mathcal{P}) = m(k-2)\frac{\pi}{k} + \varepsilon,$$

where ε is the angle, required to cover the well and approaches 0. For a large k and $m = k$, we obtain a duality gap of

$$\lim_{k\to\infty} \frac{\mathrm{AAGP}}{\mathrm{ICP}} = \lim_{k\to\infty} \frac{k(k-2)\frac{\pi}{k}}{(k-2)\frac{\pi}{k} + 2(k-1)\arcsin\left(\frac{1}{3}\right)} = \frac{\pi}{2\arcsin\left(\frac{1}{3}\right)} \approx 4.622.$$

By allowing holes within the polygon, this gap can be arbitrarily large, since an arbitrary large number of wells can connect the subpolygons.

5.3. The Full Angle Floodlight Problem

In this thesis, we discussed the AAGP. One could pose different questions such as: What is the smallest total floodlight angle to cover a polygon, if the floodlight angle has to match the internal angle of the vertex. In this section, we introduce this variant of the floodlight problem, which has similarities to both, the fundamental AGP and the AAGP. As in the AGP, the covering angle of a vertex floodlight is always the internal angle of the corner, it is placed on. The difference to the AGP is that the question is not to minimize the number of guards, but to minimize the total floodlight angle, equal to the question in the AAGP. Formally, we introduce the problem as follows. $\measuredangle_v$ describes the internal angle of a vertex v of the polygon.

Full Angle Floodlight Problem (FAFP)

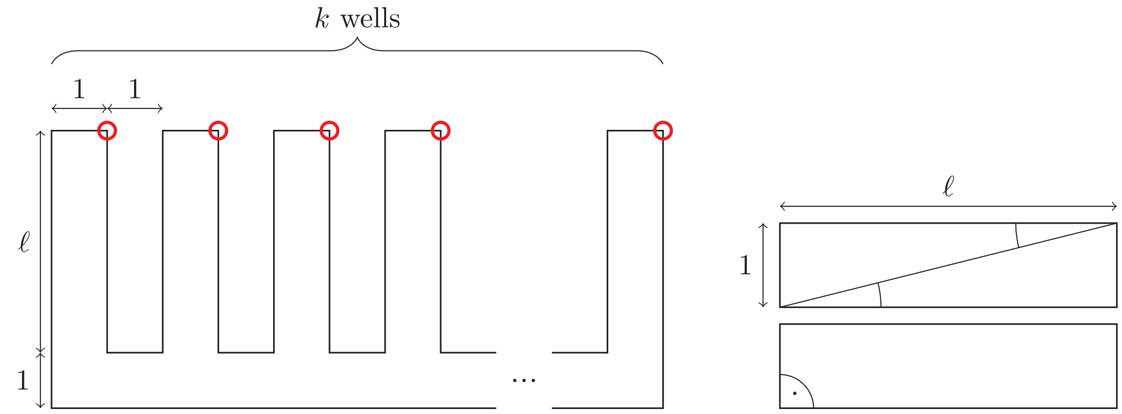
Instance: A polygon $\mathcal{P}$ with vertices V

Wanted: $\min\left\{\sum_{g\in G} \measuredangle_g \mid G \subseteq V \wedge G \text{ covers } \mathcal{P}\right\}$

In the following, we will explore the relations between the FAFP and the AGP, and the FAFP and the AAGP.

Lemma 9 (FAFP and AAGP)**.** *A solution to the FAFP on a given polygon can serve as an upper bound to solutions to the AAGP on the same polygon. The upper bound can be arbitrarily bad.*

Proof. Trivially, each FAFP solution is also a valid AAGP solution. To examine the quality of the upper bound, consider the polygon $\mathcal{P}$ given in Figure 5.6a. It is constructed from a small rectangle with k wells with height ℓ and width 1 placed on its long side. For large ℓ, no two of the red highlighted points in Figure 5.6a are visible from one vertex. Hence, we need at least k full angle floodlights with an overall guard angle of $\mathrm{FAFP}(\mathcal{P}) = k\frac{\pi}{2}$. In contrast, we can cover the polygon in the AAGP with one $\frac{\pi}{2}$-floodlight in the bottom left corner and two floodlights in each of the remaining $k-1$ wells as illustrated in Figure 5.6b. For each well, the two guards requires a total covering



(a) By increasing the number of k and substituting $\ell = k$, the total floodlight angle, required in the AAGP, is at most $\frac{\pi}{2} + 2$. In contrast, the at least required covering angle in the FAFP behaves proportionally to k.

(b) Since we require an angle of $\frac{\pi}{2}$ to cover the rectangle in the FAFP, in the AAGP an angle of $\arctan\left(\frac{1}{\ell}\right)$ is sufficient.

Figure 5.6. The FAFP as an arbitrarily bad upper bound for the AAGP.

angle of $2\arctan\left(\frac{1}{\ell}\right)$. Hence, $\mathrm{AAGP}(\mathcal{P}) \leq \frac{\pi}{2} + 2(k-1)\arctan\left(\frac{1}{\ell}\right)$. We achieve a quality ratio of

$$\frac{\mathrm{FAFP}(\mathcal{P})}{\mathrm{AAGP}(\mathcal{P})} \geq \frac{k\frac{\pi}{2}}{\frac{\pi}{2} + 2(k-1)\arctan\left(\frac{1}{\ell}\right)}.$$

For a large value of k and $\ell = k$, we obtain

$$\lim_{k\to\infty} \frac{\overbrace{k\frac{\pi}{2}}^{\to+\infty}}{\underbrace{\frac{\pi}{2} + 2(k-1)\arctan\left(\frac{1}{k}\right)}_{\to\frac{\pi}{2}+2}} \to +\infty. \qquad \square$$

Lemma 10 (FAFP and AGP)**.** *Given a polygon $\mathcal{P}$, consider optimal solutions to the FAFP and the AGP. Let n_{FAFP} and n_{AGP} be the number of required guards in these solutions and δ_{FAFP} and δ_{AGP} the required total floodlight angles. The following is always true: $n_{AGP} \leq n_{FAFP}$ and $\delta_{FAFP} \leq \delta_{AGP}$. Both upper bounds can be arbitrarily bad.*

Proof. A solution of the AGP also provides a solution of the FAFP with value δ_{AGP}. Vice versa, a solution of the FAFP provides a solution of the AGP with value n_{FAFP}.
(1) *AGP as an upper bound for FAFP*
Consider the polygon $\mathcal{P}$ given in Figure 5.7. There is no combination of three of the $2k+1$ points highlighted in red in $\mathcal{P}$ that is visible from the same vertex. Hence, one guard can covers at most two of these points and at least $k+1$ vertex guards are required to cover $\mathcal{P}$ completely as an AGP instance. There are different solutions to cover $\mathcal{P}$ with $k+1$ vertices. Thus, we consider the solution with the smallest total floodlight angle δ_{AGP}, consisting of guards placed on $x, w_1', v_2, \ldots, v_{k-1}$ and v_k. Hence, $\delta_{AGP} \geq (k-1)\beta + \gamma + \alpha$.

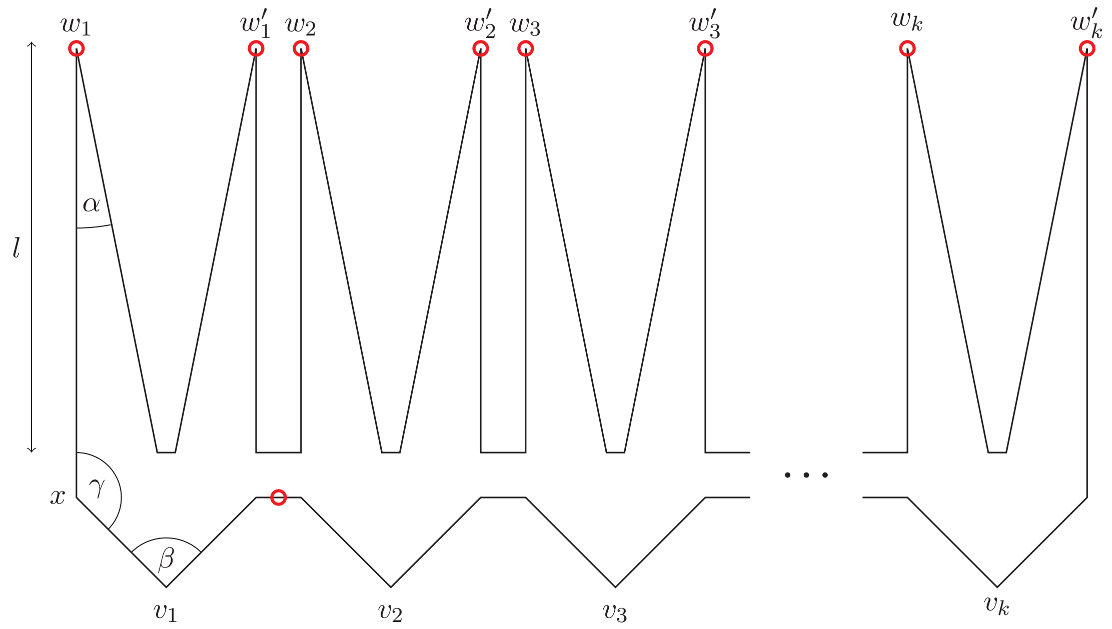


Figure 5.7. No three of the red highlighted points are visible by the same vertex. Hence, a minimal covering in the AGP consists of $k+1$ guards, placed on $x, v_2, \ldots, v_k$. In the FAFP, guards on $w_1, w_1', \ldots, w_k, w_k'$ provides an optimal solution.

In contrast, a solution of the FAFP with an angle $\leq 2k\alpha + \gamma$ can be determined by placing guards on $w_1, w_1', \ldots, w_k, w_k'$ and x. We obtain a quality ratio of

$$\frac{\delta_{\text{AGP}}}{\delta_{\text{FAFP}}} \geq \frac{(k-1)\beta + \gamma + \alpha}{2k\alpha + \gamma} = \frac{k\beta + \gamma}{2k \arctan\left(\frac{1}{\ell}\right) + \gamma}.$$

For a large k and by substituting $\ell = k$ we obtain

$$\lim_{k\to\infty} \frac{\overbrace{(k-1)\beta + \gamma + \alpha}^{\to+\infty}}{\underbrace{2k \arctan\left(\frac{1}{k}\right) + \gamma}_{\to 2+\gamma}} \to +\infty.$$

(2) *FAFP as an upper bound for AGP*
Consider the star shaped polygon $\mathcal{P}$ as given in Figure 5.7. It is constructed as a star with an odd number of k prongs, but one prong is directed inside the polygon to the center m of the star. Hence, one vertex guard placed on m is sufficient to cover the entire polygon. Using the fact that k is odd, we can determine two characteristics of the star polygon. On the one hand, from an outer vertex v_i, no other outer vertex v_j is visible, if ℓ is large enough. On the other hand, for a large ℓ, only three outer vertices are visible from an inner vertex w_i, namely the adjacent vertices v_i and v_{i+1} and the vertex on the opposite side of m. Following, there are three kinds to cover the outer vertices $v_1, \ldots, v_{k-1}$:

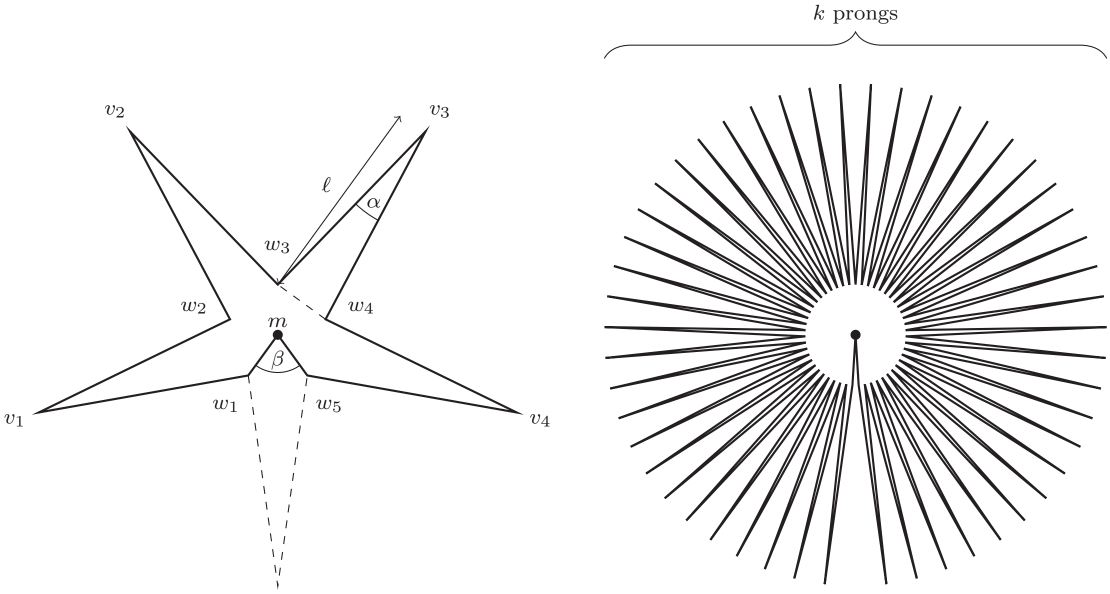


Figure 5.8. If k is the number of prongs, we obtain $\beta = \frac{2\pi}{k}$. By increasing ℓ, α approaches 0, but β stays constant. For a large k and a large ℓ, the required number of guards in the AGP is one, placed on m. An optimal solution in the FAFP consists of $k-1$ guards with an angle of α each.

1. Cover three outer vertices with a guard on an inner vertex, whose angle is $\frac{(n-2)\pi}{2n} - \alpha$,
2. cover all outer vertices with one guard placed on m with an angle of $2\pi - \beta$ and
3. cover one outer vertex with a guard placed on it with an angle α.

Since β stays the same, when increasing the length ℓ, the angle α converges to the limiting value 0. Hence, for a large l, an optimal solution of the FAFP consists of $k-1$ guards, placed on the outer vertices $v_1, \ldots, v_{k-1}$. We obtain a quality ratio of

$$\frac{n_{\text{FAFP}}}{n_{\text{AGP}}} = \frac{k-1}{1} = k-1.$$

For a large k, the limiting value converges to

$$\lim_{k\to\infty} k-1 \to +\infty. \qquad \square$$

6 Conclusion

Within this thesis, we have introduced a new variant of the floodlight problem, called the Angular Art Gallery Problem. The angle of a floodlight in the AAGP is not fixed to a constant value, but may be set individually for each floodlight. The AAGP asks for a set of floodlights with a minimum total floodlight angle, covering a given polygon completely.

We have presented three main results. Initially, we have proven an upper bound of

$$(n-1)\frac{\pi}{6} \qquad \textit{(Histograms)}$$

for histograms with n vertices. An iterative algorithm has been presented, determining an upper bound covering for histograms by iterating over the peak edges and covering the area between the baseline and an edge with an angle of at most $\frac{\pi}{6}$ per edge. This bound has been proven to be worst case optimal by using the successive result for equilateral triangles. Secondly, we have proven that an angle of

$$\frac{\pi}{3} \qquad \textit{(Equilateral Triangles)}$$

is required to cover an equilateral triangle with a finite number of vertex floodlights. Given an arbitrary covering of the triangle, we can show that the summed up angle of this covering is not smaller than this lower bound. Finally, we have given a general upper bound for simple polygons. A total floodlight angle of at most

$$(n-2)\frac{\pi}{4} \qquad \textit{(Simple Polygons)}$$

is always sufficient to cover a given simple polygon with n vertices. The presented algorithm uses a triangulation of the polygon to partitions this triangulation in sets of adjacent triangles of a certain size. Afterwards, each new formed subpolygon is covered independently, resulting in a solution, satisfying the proposed upper bound.

Future Work In the previous section, multiple objects of further research have already been presented. In particular, the upper bound for simple polygons may be reduced by proving a bound of $(n-1)\frac{\pi}{6}$ for simple polygons of limited size. Thus, proving this bound for simple polygons with $n \leq 21$ vertices would suffice to reduce the general bound for simple polygons to 1.1-times the assumed tight bound of $(n-1)\frac{\pi}{6}$. The introduced FAFP provides a wide range of new questions, including upper and lower bounds. The NP-hardness of the FAFP can be shown by applying a 3-SAT reduction, similar to the one used in the hardness proof of the α-Floodlight Problem [3]. The literal and clause gadgets can be adapted, such that certain guards are forced in an optimal solution.

Besides, a generalization of the theorem for equilateral triangles to general triangles can possibly be proven with methods, similar to the methods used here. First experiments suggest that several of the results for equilateral triangles hold for general triangles, too. Furthermore, a class of polygons, not examined in this thesis, are orthogonal polygons. An upper bound of $\lfloor \frac{n}{4} \rfloor \frac{\pi}{2}$ could be proven by partitioning an orthogonal polygon into L-shaped subpolygons and covering them independently, like shown in [25]. Finally, we will complete this thesis like we introduced it: Solving the initial question about optimal covering of rectangles and especially squares would help to construct a lower bound example which would show that the upper bound for orthogonal polygons is tight.

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Appendices

A Histogram Covering Algorithm

A formal definition of the algorithm, introduced in Chapter 2, is given.

Algorithm 1 $\frac{\pi}{6}$ – histogram covering

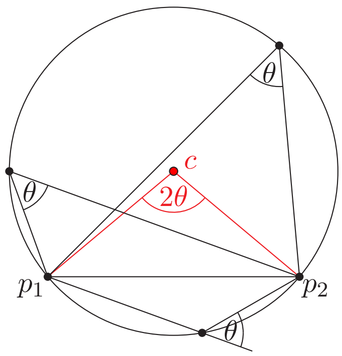
Input A histogram $\mathcal{P}$ with n vertices
Output A feasible point floodlight covering $\mathcal{C}$ for $\mathcal{P}$ with $\measuredangle_{\mathcal{C}} \leq (n-1)\frac{\pi}{6}$

$i \leftarrow 0$
$\mathcal{C} = \emptyset$
while $i \neq n-1$ **do**
    if not $e_i = 1$ is covered like required in case 1 **then** ▷ Case 1
        $fp \leftarrow$ intersection$(VE_{\text{left}}(e_i), s)$
        $\delta_0 \leftarrow$ intersection angle of $VE_{\text{left}}(e_i)$ and s
        if v_{i+1} is reflex **then** ▷ Case 2
            $\mathcal{C} \leftarrow \mathcal{C} \cup \{(fp, 0, \delta_0)\}$
            $\delta_1 \leftarrow$ intersection angle of e_i and e_{i+1}
            $\mathcal{C} \leftarrow \mathcal{C} \cup \{(v_{i+1}, \delta_0, \delta_0 + \delta_1)\}$
            if v_{i+2} is reflex **then**
                $\delta_2 \leftarrow$ intersection angle of e_{i+1} and e_{i+2}
                $\mathcal{C} \leftarrow \mathcal{C} \cup \{(v_{i+2}, \delta_1, \delta_1 + \delta_2)\}$
        else if v_{i+2} is convex **then** ▷ Case 3
            $\mathcal{C} \leftarrow \{(fp, 0, \delta_0)\}$
        else if $\exists\, j : i+1 < j < n : e_j$ is **not** covered ▷ Case 4
                        and v_j is reflex
                        and e_j is visible from v_i **then**
            $\mathcal{C} \leftarrow \mathcal{C} \cup \{(\text{intersection}(VE_{\text{left}}(e_j), s), 0, \delta_0)\}$
        else if $VE_{\text{right}}(e_{i+1})$ intersects with s **then** ▷ Case 5
            Let f be the floodlight with the smallest angle that covers the resulting triangle defined by e_i, e_{i+1} and s
            $\mathcal{C} \leftarrow \mathcal{C} \cup f$
        else ▷ Case 6
            $fp \leftarrow (x$ coordinate of the right vertex of $VE_{\text{right}}(e_i), 0)$
            $\mathcal{C} \leftarrow \mathcal{C} \cup \{(fp, \frac{\pi}{2}, \pi)\}$
            $i \leftarrow i + 1$
    $i \leftarrow i + 1$
return $\mathcal{C}$

B Equilateral Triangles

B.1. Inscribed Angle Theorem

Theorem (Inscribed Angle Theorem)**.** *Given a circle with center c and three points p_1, p_2 and q, lying on the circle. Let θ be the intersection angle of $\overline{p_1q}$ and $\overline{p_2q}$. The relation $\measuredangle p_1cp_2 = 2\theta$ holds.*
As a result, the angle θ stays the same, regardless of the point q.



B.2. Properties of $\beta_0(y)$ and $\gamma_0(y)$

Consider the functions β_0 and γ_0 defined in Chapter 3:

$$\beta_0(y) = \arctan\left(\frac{y}{\frac{1}{2} - \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)}\right)$$
$$\gamma_0(y) = \arctan\left(\frac{y}{\frac{1}{2} + \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)}\right).$$

We show that these functions fulfill the following properties:

1. $\beta_0(y) \geq \gamma_0(y)$ for all $0 \leq y \leq \frac{\sqrt{3}}{2}$
2. The gradients of β_0 and γ_0 have a unique intersection within the relevant domain
3. At 0, the gradient of γ_0 is smaller than the gradient of β_0
4. At $\frac{\sqrt{3}}{2}$, the gradient of β_0 is smaller than the gradient of γ_0

With respect to the first property, consider the triangles T_1 and T_2 in Figure B.1. Since the line segment s is on the left side of the perpendicular bisector, the adjacent side l_1 of β_0 is smaller than or equal to the adjacent side l_2 of γ_0 for all α and y. The opposite side of both β_0 and γ_0 is y. According to the trigonometric functions the angle β_0 is always greater than or equal to γ_0.

The remaining properties are shown by evaluation of $\beta_0\prime$ and $\gamma_0\prime$ at -1, 0 and $\frac{\sqrt{3}}{2}$. That results in $\beta_0\prime(-1) \leq \gamma_0\prime(-1)$, $\beta_0\prime(0) \geq \gamma_0\prime(0)$ and $\beta_0\prime(\frac{\sqrt{3}}{2}) \leq \gamma_0\prime(\frac{\sqrt{3}}{2})$. Hence one intersection is in the range $[-1, 0)$, which is not relevant, since we consider only positive y values. A second intersection is in the range $[0, \frac{\sqrt{3}}{2}]$. The derivations of β_0 and γ_0 are

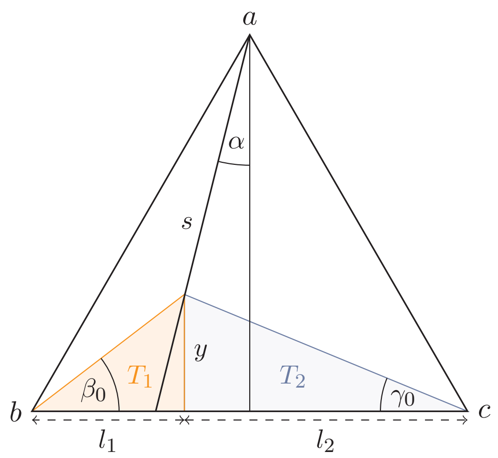


Figure B.1. Since $l_1 \leq l_2$ for all α and all y, it follows $\beta_0 \geq \gamma_0$.

of the form $\frac{d}{ax^2+bx+c}$. Hence, their intersections can be determined with the quadratic formula and the number of intersections is limited to 2.

B.3. Global Minimum of $\beta(y)+\gamma(y)$

Consider the functions β (Equation 3.2) and γ (Equation 3.3) given in Chapter 3:

$$\beta(y) = \arctan\left(\frac{\frac{1}{2} - \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)}{y}\right) - \frac{\pi}{6}$$

$$\gamma(y) = \arctan\left(\frac{y}{\frac{1}{2} + \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)}\right)$$

Their respective first derivations are

$$\frac{d}{dy}\beta(y) = \frac{\sqrt{3}\tan(\alpha) - 1}{2y^2 + 2\left(\frac{1}{2} - \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right)\right)^2}$$

$$\frac{d}{dy}\gamma(y) = \frac{\tan(\alpha)y + \tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right) + \frac{1}{2}}{y^2 + \left(\tan(\alpha)\left(\frac{\sqrt{3}}{2} - y\right) + \frac{1}{2}\right)^2}.$$

We can determine the extreme values of $\beta(y)+\gamma(y)$ by setting the sum of its derivations $\frac{d}{dy}\beta(y)+\frac{d}{dy}\gamma(y)$ equal to 0:

$$
\begin{aligned}
&0 = \frac{d}{dy}\beta(y) + \frac{d}{dy}\gamma(y) \\
\iff &0 = \frac{\sqrt{3}\tan(\alpha)-1}{2y^2+2\left(\frac{1}{2}-\tan(\alpha)\left(\frac{\sqrt{3}}{2}-y\right)\right)^2} + \frac{\tan(\alpha)y+\tan(\alpha)\left(\frac{\sqrt{3}}{2}-y\right)+\frac{1}{2}}{y^2+(\tan(\alpha)\left(\frac{\sqrt{3}}{2}-y\right)+\frac{1}{2})^2} \\
\iff &0 = \left(\sqrt{3}tan(\alpha)-1\right)\left(y^2+\left(\tan(\alpha)\left(\frac{\sqrt{3}}{2}-y\right)+\frac{1}{2}\right)^2\right) \\
&\quad + \left(\tan(\alpha)y+\tan(\alpha)\left(\frac{\sqrt{3}}{2}-y\right)+\frac{1}{2}\right)\left(2y^2+2\left(\frac{1}{2}-\tan(\alpha)\left(\frac{\sqrt{3}}{2}-y\right)\right)^2\right) \\
\iff &0 = 2\sqrt{3}\tan(\alpha)\left(\tan(\alpha)^2+1\right)y^2+2\tan(\alpha)\left(1-3\tan(\alpha)^2\right)y \\
&\quad - \frac{\sqrt{3}}{2}\tan(\alpha)\left(3\tan(\alpha)^2-1\right) \\
\iff &0 = y^2+\frac{1-3\tan(\alpha)^2}{\sqrt{3}\left(\tan(\alpha)^2+1\right)}y+\frac{3\tan(\alpha)^2-1}{4\left(\tan(\alpha)^2+1\right)} \\
\iff &y = -\frac{1-3\tan^2(\alpha)}{2\sqrt{3}\left(\tan^2(\alpha)+1\right)} \pm \sqrt{\left(\frac{1-3\tan^2(\alpha)}{2\sqrt{3}\tan^2(\alpha)+2}\right)^2-\frac{3\tan^2(\alpha)-1}{4\left(\tan^2(\alpha)+1\right)}}
\end{aligned}
$$

To find the global minimum in the domain $\left(0,\frac{\sqrt{3}}{2}\right)$ we have to prove three more things:

- Only one of the two values is within the domain:

$$
y = \underbrace{-\frac{1-3\tan^2(\alpha)}{2\sqrt{3}\tan^2(\alpha)+2}}_{<0 \text{ for all } \alpha\in\left(0,\frac{\pi}{6}\right)} \pm \underbrace{\sqrt{\left(\frac{1-3\tan^2(\alpha)}{2\sqrt{3}\tan^2(\alpha)+2}\right)^2-\frac{3\tan^2(\alpha)-1}{4\left(\tan^2(\alpha)+1\right)}}}_{\geq 0 \text{ for all } \alpha\ \in\left(0,\frac{\pi}{6}\right)}
$$

  The second zero value $(-)$ is negative, so it can be ignored.

- The values at the boundaries 0 and $\frac{\sqrt{3}}{2}$ are greater than the local minimum:
  Evaluation of $\beta(y) + \gamma(y)$ at the boundaries yields

$$\beta(y \to 0) = \lim_{y \to 0} \arctan\left(\frac{\frac{1}{2} - \tan(\alpha)}{y}\right) - \frac{\pi}{6} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

$$\gamma(0) = \arctan(0) = 0$$

$$\beta\left(\frac{\sqrt{3}}{2}\right) = \arctan\left(\frac{1}{2} \cdot \frac{2}{\sqrt{3}}\right) - \frac{\pi}{6} = 0$$

$$\gamma\left(\frac{\sqrt{3}}{2}\right) = \arctan\left(\frac{\sqrt{3}}{2} \cdot \frac{2}{1}\right) = \frac{\pi}{3}$$

  This implies a function value of $\frac{\pi}{3}$ at the domain boundaries. Section B.4 discusses that the extreme value is always on an arc. At $\alpha = 0$ the value of $\beta + \gamma$ is $\frac{\pi}{3}$. For all $\alpha > 0$ the value of $\beta + \gamma$ is smaller than $\frac{\pi}{3}$.

- The relevant value is a local minimum:
  This is a direct result of the previous result. If the extreme value would be a maximum, there would be at least two more extreme values between the boundaries and the extreme value.

B.4. Intersection of Arc c and Line Segment s

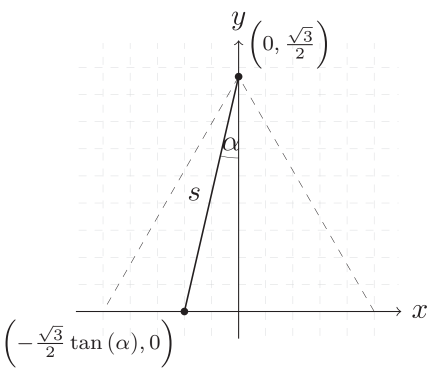


Figure B.2. The upper end point of the line segment s is defined by the height of the triangle, the lower end point can be determined by applying the trigonometric functions.

Given the line segment s defined by α, we can determine the corresponding line g with $g(x) = mx + b$ through the endpoints of s (see Figure B.2).

$$\begin{aligned} g(0) = \frac{\sqrt{3}}{2} &= m * 0 + b \\ \rightarrow b &= \frac{\sqrt{3}}{2} \\ g\left(-\frac{\sqrt{3}}{2}\tan(\alpha)\right) = 0 &= m * -\frac{\sqrt{3}}{2}\tan(\alpha) + \frac{\sqrt{3}}{2} \\ \rightarrow m &= \frac{1}{\tan(\alpha)} \end{aligned}$$

Next, we determine the intersection of g and the circle c defined in Equation 3.1.

$$c : \frac{1}{3} = x^2 + \left(y + \frac{1}{2\sqrt{3}}\right)^2 \tag{B.1}$$

$$g : y = \frac{x}{\tan(\alpha)} + \frac{\sqrt{3}}{2}$$

$$\iff x = \tan(\alpha)\left(y - \frac{\sqrt{3}}{2}\right) \tag{B.2}$$

By substituting Equation B.2 into Equation B.1 we obtain the intersection point:

$$\begin{aligned} &\frac{1}{3} = \tan^2(\alpha)\left(y - \frac{\sqrt{3}}{2}\right)^2 + \left(y + \frac{1}{2\sqrt{3}}\right)^2 \\ \iff &\frac{1}{3} = \tan^2(\alpha)\left(y^2 - \sqrt{3}y + \frac{3}{4}\right) + \left(y^2 + \frac{y}{\sqrt{3}} + \frac{1}{12}\right) \\ \iff &\frac{1}{3} = \left(\tan^2(\alpha) + 1\right)y^2 + \left(\frac{1}{\sqrt{3}} - \tan^2(\alpha)\sqrt{3}\right)y + \frac{3}{4}\tan^2(\alpha) + \frac{1}{12} \\ \iff &0 = y^2 + \frac{\frac{1}{\sqrt{3}} - \sqrt{3}\tan^2(\alpha)}{\tan^2(\alpha) + 1}y + \frac{\frac{3}{4}\tan^2(\alpha) - \frac{3}{12}}{\tan^2(\alpha) + 1} \\ \iff &0 = y^2 + \frac{1 - 3\tan^2(\alpha)}{\sqrt{3}\left(\tan^2(\alpha) + 1\right)}y + \frac{3\tan^2(\alpha) - 1}{4(\tan^2(\alpha) + 1)} \\ \iff &y = -\frac{1 - 3\tan^2(\alpha)}{2\sqrt{3}\tan^2(\alpha) + 2} \pm \sqrt{\left(\frac{1 - 3\tan^2(\alpha)}{2\sqrt{3}\tan^2(\alpha) + 2}\right)^2 - \frac{3\tan^2(\alpha) - 1}{4\left(\tan^2(\alpha) + 1\right)}} \end{aligned}$$

This value equals to the value determined for the global minimum of the function $\beta(y) + \gamma(y)$ determined in Section B.3.